Universität Duisburg-Essen
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DISSERTATION

Aspects of Spectral Statistics in the Correlated Wishart Model

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Hiermit versichere ich, die vorliegende Dissertation selbstständig, ohne fremde Hilfe und ohne Benutzung anderer als den angegebenen Quellen angefertigt zu haben. Alle aus fremden Werken direkt oder indirekt übernommenen Stellen sind als solche gekennzeichnet. Die vorliegende Dissertation wurde in keinem anderen Promotionsverfahren eingereicht. Mit dieser Arbeit strebe ich die Erlangung des akademischen Grades "Doktor der Naturwissenschaften" (Dr. rer. nat.) an.

Datum

Tim Wirtz

*To Sara for all her encouragement,
her patience and her love.*

Zusammenfassung

Wenn komplexe Systeme unter Zuhilfenahme von multivariaten, empirischen Zeitreihen untersucht werden, spielt die Korrelationsmatrix und ihre Eigenwerte eine zentrale Rolle, weil sie wichtige Information über die Dynamik des Systems beinhalten. In der Regel ist die Anzahl der Zeitschritte in den Zeitreihen eher kurz, so dass die Korrelationsmatrix selbst zu einer fluktuierenden Größe wird. Um die empirisch bestimmte Korrelationsmatrix bewerten zu können, wird diese mit einer Nullhypothese verglichen. Unter der Annahme von gaußscher Statistik ergibt sich als Nullhypothese das korrelierte Wishart-Modell.

In der vorliegenden Arbeit beschäftige ich mich mit der Spektralstatistik innerhalb des korrelierten Wishart-Modells, erweitere die Supersymmetrie-Methode, wende sie auf Fragestellungen der Spektralstatistik an und entwickele einen neuen Ansatz, um die Statistik des kleinsten und des größten Eigenwertes analysieren zu können. Auf Grund ihrer systemspezifischen Relevanz, liegt mein Hauptaugenmerk auf den extremen Eigenwerten. Des Weiteren beschäftige ich mich mit der Statistik des Gesamtspektrums der Wishart Korrelationsmatrizen.

In den ersten beiden Kapiteln begründe ich den von mir verwendeten Ansatz und fasse die theoretischen Grundlagen dieser Arbeit zusammen. Als erstes Ergebnis dieser Arbeit erweitere ich die Anwendbarkeit der allgemeinen Hubbard-Stratonovich Transformation auf korrelierte Wishart-Modelle.

Im Anschluss daran untersuche ich die Verteilung des kleinsten Eigenwertes im nicht-korrelierten, reellen und im nicht-korrelierten, reell-quaternionischen Wishart-Modell. Für Ersteres leite ich im Falle endlicher Matrixdimensionen und im mikroskopischen Limes kompakte Ausdrücke her. Im reell-quaternionischen Modell gelingt mir die Konstruktion von pfaffschen Strukturen für die Verteilung des kleinsten Eigenwertes.

Im vierten Kapitel wende ich die Methode der Supersymmetrie auf das korrelierte Wishart- und Jacobi-Modell an. Zunächst untersuche ich im Wishart-Modell die Statistik der extremen Eigenwerte und konstruiere für die dazugehörigen Wahrscheinlichkeiten bis dato unbekannte invariante Matrixmodelle. Im Falle endlicher Matrixdimensionen und im mikroskopischen Limes leite ich für die Verteilung des kleinsten Eigenwertes einen kompakten Ausdruck her. In der Folge berechne ich im reellen und im komplexen, korrelierten Jacobi-Modell die Eigenwertdichte und finde für das reelle Ensemble ein Zweifachintegral- und für das komplexe Ensemble einen exakten Ausdruck. Zwischen zwei Wishart-Modellen, mit und ohne Entartung in den empirischen Eigenwerten, leite ich im Anschluss einen asymptotischen Zusammenhang her und zeige, dass sich deren spektrale Statistik universell verhält.

Im letzten Kapitel dieser Arbeit entwickele ich einen neuen Ansatz zur Untersuchung der Spektralstatistik in korrelierten Wishart-Modellen. Dieser erlaubt es, das Itzykson-Zuber Integral zu umgehen und Eigenwertintegrale mit Standardmethoden zu untersuchen. Angewandt auf die Statistik des größten Eigenwertes leite ich im Anschluss für dessen kumulative Dichtefunktion einen exakten Ausdruck her. Abschließend zeige ich, dass der kleinste und der größte Eigenwert in einigen Fällen Tracy-Widom verteilt ist.

Abstract

When multivariate empirical time series are considered to study complex systems the correlation matrix and its eigenvalues play a central role, because they bear rich information about the dynamics of the system. In the majority of the applications of time series analysis, the length of the time series is rather short such that the correlation matrix inherits statistical fluctuations. To quantify the significance of the empirically estimated correlation matrix, it is compared to a null hypothesis. Under the assumption of Gaussian statistics, the null hypothesis turns out to be the correlated Wishart model.

I study aspects of the spectral statistics in the correlated Wishart model, extend and apply the method of supersymmetry to it and develop a new approach to study the statistics of the smallest and the largest eigenvalue. I focus mainly on the extreme eigenvalues, because they carry significant system specific information. In addition I consider the statistics of the eigenvalues in the bulk.

In the first two parts of this thesis I briefly motivate my approach from the physical point of view and summarize the tools and statistical quantities important later. As a first result of this thesis I extend the generalized Hubbard-Stratonovich transformation to include also correlated Wishart ensembles.

In the third part I am concerned with the distribution of the smallest eigenvalue within the real and the real quaternion uncorrelated Wishart model. For the real ensemble with even rectangularity, I derive an exact expression for the distribution as well as its microscopic limit and for the real quaternion model I uncover a Pfaffian structure.

In the fourth part I apply the method of supersymmetry to eigenvalue statistics of the more involved correlated Wishart and Jacobi ensemble. I study the statistics of the extreme eigenvalues in the former and derive for both quantities previously unknown invariant matrix models. I calculate an exact expression and the microscopic limit of the smallest eigenvalue distribution. In the correlated Jacobi model I compute using supersymmetry the level density and obtain for the real ensemble a twofold integral expression and for the complex ensemble a closed-form expression. After this interlude I derive an asymptotic relation between bulk eigenvalue statistics of two real Wishart models with and without a degeneracy in the empirical eigenvalues and show that the local eigenvalue fluctuations are universal.

In the fifth part I develop a new approach to analyze the eigenvalue statistics in the correlated Wishart model by circumventing the Itzykson-Zuber integral. I apply it to the gap probabilities related to the smallest and the largest eigenvalue distribution and derive an exact expression for the cumulative density function of the latter. Finally, I show that in some cases the smallest and the largest eigenvalue are Tracy-Widom distributed.

Contents

Publications

Parts of this thesis are included in the following publications and manuscripts:

[1] T. Wirtz and T. Guhr, “Distribution of the Smallest Eigenvalue in the Correlated Wishart Model”, Phys. Rev. Lett., vol. 111, p. 094101, 2013

[2] T. Wirtz and T. Guhr, “Distribution of the smallest eigenvalue in complex and real correlated Wishart ensembles”, J. Phys. A, vol. 47, no.7, p. 075004, 2014

[3] G. Akemann, T. Guhr, M. Kieburg, R. Wegner, and T. Wirtz, “Completing the Picture for the Smallest Eigenvalue of Real Wishart Matrices”, Phys. Rev. Lett., vol. 113, p. 250201, 2014

[4] T. Wirtz, M. Kieburg and T. Guhr “Limiting Statistics of the Largest and Smallest Eigenvalues in the Correlated Wishart Model” EPL, vol. 109, no. 2 , p. 20005, 2015

[5] T. Wirtz, G. Akemann, T. Guhr, M. Kieburg and R. Wegner, “The Smallest Eigenvalue Distribution in the Real Wishart-Laguerre Ensemble with Even Topology”, arXiv:math-ph/1502.03685 (2015), submitted to J. Phys. A.

[6] T. Wirtz, M. Kieburg and T. Guhr, “Asymptotic Relation Between the Statistics of Degenerate and Non-Degenerate Wishart Ensembles”, to be submitted to Phys. Rev. Lett. 2015

[7] T. Wirtz, D. Waltner, M. Kieburg and S. Kumar, “Supersymmetry for the k-point Generating Function of the Correlated Jacobi Model: Exact Results”, in preparation 2015

The following publication is not part of this thesis:

[8] D. Waltner, T. Wirtz and T. Guhr, “Eigenvalue Density of the Doubly Correlated Wishart Model: Exact Results”, arXiv:math-ph/1412.3092, accepted for publication in J. Phys. A in March 2015

Author Contributions

Here, I lay out my contributions to the publications and manuscripts mentioned above:

[1] The letter reports on the derivation of an exact expression and a universality for the smallest eigenvalue distribution in the real and the complex correlated Wishart model. The project was supervised by T. Guhr. I performed all the calculations and did the numerical simulations. The text was partially written by T. Guhr and by me.

[2] The paper extends reference [1] with a discussion of previously unknown dualities, an extension of the universality and a detailed derivation of all results and of the microscopic limiting expressions. The text was written by me.

[3] The smallest eigenvalue distribution and a related gap probability are computed in the real uncorrelated Wishart model with even rectangularity. All calculations and simulations were mainly done by me in collaboration with the other authors. The project was initiated by G. Akemann. It was supervised by G. Akemann and T. Guhr. The text was written by all authors based on a first version by G. Akemann.

[4] The letter introduces the Fourier approach and shows that the smallest and largest eigenvalue is in some cases Tracy-Widom distributed. The project was supervised by T. Guhr. I did all calculations, numerical simulations, discovered the Tracy-Widom distribution and extended an idea of M. Kieburg yielding the Fourier approach. The text was written by myself.

[5] The paper extends reference [3] with a more general construction of the polynomials and a variety of additional results. All calculations and simulations were basically done by me in collaboration with the other authors. The text writing was mainly done by myself.

[6] The letter reports on a discovery of an asymptotic relation between two Wishart models with and without degeneracy in the empirical eigenvalues. It was supervised by T. Guhr. The idea to this project emerged in a discussion of all three authors. I performed the detailed analysis and did all numerical simulations. The text was basically written by myself.

[7] The paper is concerned with the eigenvalue statistics of the real and the complex correlated Jacobi model. I did all computations, except the calculations in the supermatrix model to derive the final expressions and the numerical simulations, which I did in collaboration with D. Waltner. The text was partially written by D. Waltner and by me.

[8] The derivation of the level density in the doubly correlated Wishart model is shown. It was initiated and supervised by T. Guhr. The calculations were mainly done by D. Waltner. I contributed to the saddle point approximation and the numerical simulations. The text was written by D. Waltner.

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First of all I would like to say thanks to Sara. Her never ending patience, her love and her continous support encouraged me to do my PhD and finally to write this thesis.

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CHAPTER 1

Introduction

Our intention is to model and analyze the correlations in empirical time series on a general level. Time series carry rich information about the dynamics; to a large extend this information is reflected in the correlations. Although the considered systems behave causal and deterministic microscopically, on a coarse-grained scale they behave statistically. In good agreement with empirical studies, the fluctuations in the observed or measured time series are modeled by random variables. Since in many cases these systems have beside their complexity some structure, the observed data is not completely random but shows correlations within the current state or observation. In this thesis we approach the correlations by an ensemble approach, which we discuss in the upcoming sections in more detail.

Before addressing the methodology, we introduce three common examples of complex systems where time series analysis applies. Electroencephalography (EEG) as introduced by Hans Berger in 1924, is a medical procedure to measure the brain waves of a subject [9]. A number of electrodes are attached to the scalp of a subject to record the fluctuations of brain electric potentials. The signals are caused by cortical synaptic actions in the brain's outer layers and change in the 10 to 100 milliseconds range. On the left of Fig. 1.1, we show a test person wearing many electrodes and on the right the recording of an electroencephalogram. These signals are recorded for research as well as for clinical use. A common application is the detection of epilepsy [11]. If an epileptic seizure happens the brain waves are strongly correlated over large regions of the brain.

The weather, our second example, is driven by dynamical changes and never reaches any stationary state. Because of this non-stationarity and its complexity it avoids any precise long term prediction of its future state. It possesses many observables like relative temperature and pressure differences, the wind velocity and direction, etc., each of which is assigned to a spot on the globe. On their own, these observables seem to be randomly distributed. Taken as a whole, it turns out that the system is structured not only in space but also in time. We emphasize this in Fig. 1.2, showing the current state of the US weather (on 16th of February 2015) in terms of its high and low pressure-regions and the same picture six hours later. Although, the low and high pressure-regions seem to be randomly distributed, they cluster locally . Moreover, comparing the current and the historical state, the

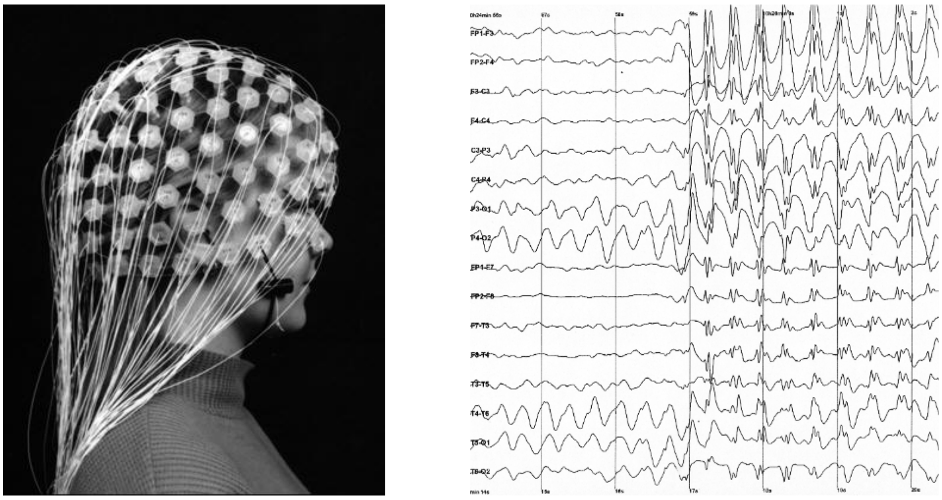


Figure 1.1: Left: A patient prepared for electroencephalography. Taken from Ref. [10]. Right: Recording of brain waves using sixteen electrodes. Taken form the Wikipedia entry "Electroencephalography".

evolution of the high and low pressure-regions influence themselves mutually. In a laboratory, the evolution of a current state of the weather can be modeled using the Navier-Stokes equation. Because of the complexity small changes in the initial data or the current state, change drastically the future state. This yields a non-negligible random component.

The financial market with its various branches, products and involved institutes is a complex system that does not originate in any natural science [12,13]. Similar to the brain waves, it does it provide any underlying law like the laws of thermodynamics. Its complexity and dynamics are caused by world wide trading, the psychology of the brokers, the diversity of investment strategies, current trends in the industrial branches, the historical, current and expected economic growth, etc. The outcome of the financial market are the stock prices. Analogous to the brain waves and the weather the financial market is not completely random, it possesses some structure. This can be seen at best when studying the correlations between returns [14] (the relative stock price differences). The authors showed, upon the example of the Standard & Poor 500 index (S&P500), how the correlations within the financial market are constantly changing its structure, see Fig .1.3.

1.1 Time Series Analysis and Correlations

Time series analysis has become a powerful tool when studying generic features of physical systems on a general level [15–17]. A time series is a vector X with either real, complex or real quaternion entries $X(t)$, $t = 1, \ldots, n$. For the majority of systems time series have real entries and in some exceptional cases like wireless communication they are complex. To the best of our knowledge, real quaternion time series have no applications in this context. Nevertheless, we will consider all

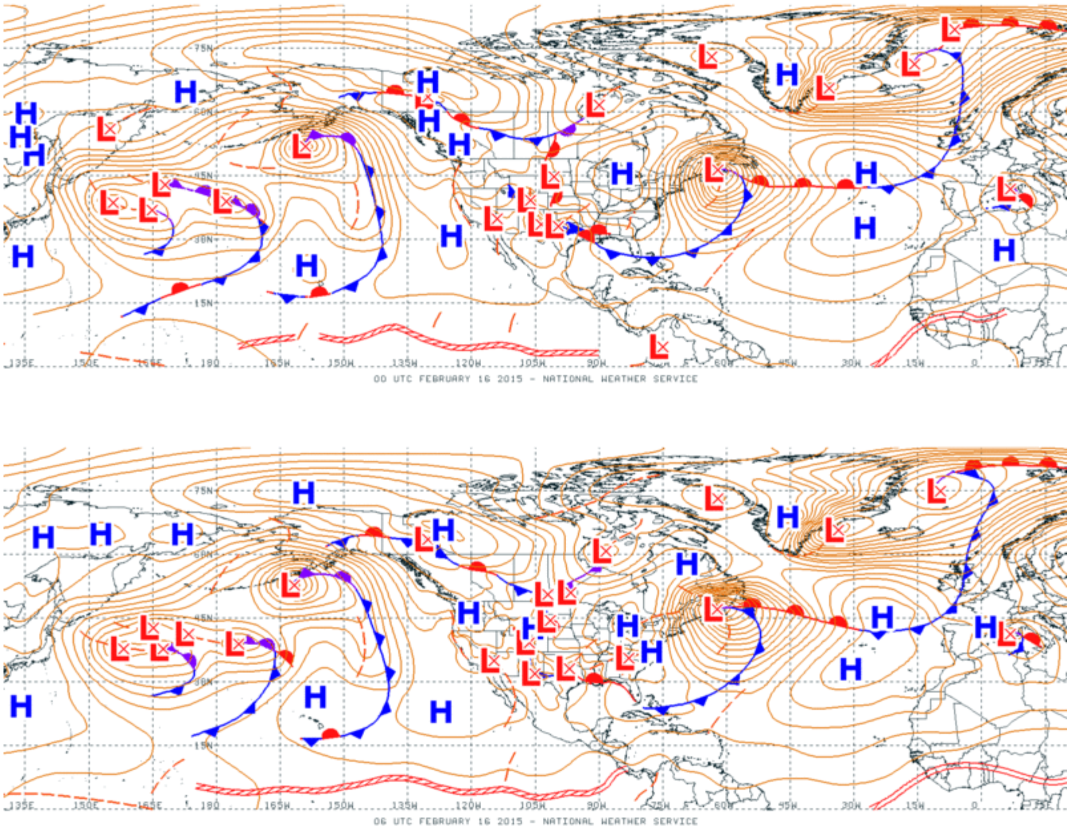


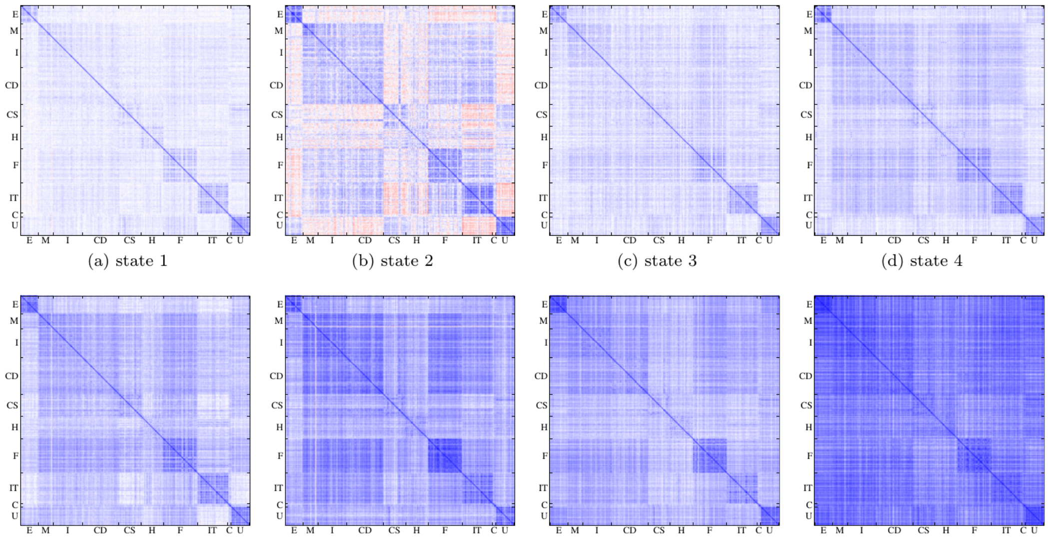
Figure 1.2: Top: Current state of the US weather (0:00 o'clock a.m. on 16th of February 2015) in terms of its high (**H**) and low (**L**) pressure-regions . Bottom: The same picture six hours later. Taken from *http://www.opc.ncep.noaa.gov/* .

three classes in a unified approach unless otherwise stated.

Generally, more than only one time series is recorded such that a set of them X_i with $i = 1 \ldots, p$ is observed. In the examples given in the previous section, the time series arise as simultaneously measured signals by the electrodes on the scalp of a subject, as the measured temperature or pressure at different locations or the stock prices in a portfolio.

A set of recorded brain wave signals is illustrated in the right plot of Fig. 1.1. In Fig. 1.4, we show time series of the temperature in Duisburg and Aachen on a hourly basis and the stock prices of Bayer AG and Addidas AG on a daily basis. All plots emphasize the rich information carried by the time series. The temperature time series clearly shows a seasonal structure. It is similar for Duisburg and Aachen which might be expected, because the distance between both cities is about 100 km only. The stock prices seem to follow a similar trend, at least for parts of the time series.

On the contrary, the plots show statistical fluctuations, which can be modeled



(a) state 1 (b) state 2 (c) state 3 (d) state 4

(e) state 5 (f) state 6 (g) state 7 (h) state 8

Figure 1.3: Non-stationarity of the finance market illustrated with the evolution of the correlations within the S&P 500 index over the period 1990-2010. Taken from Ref. [14]

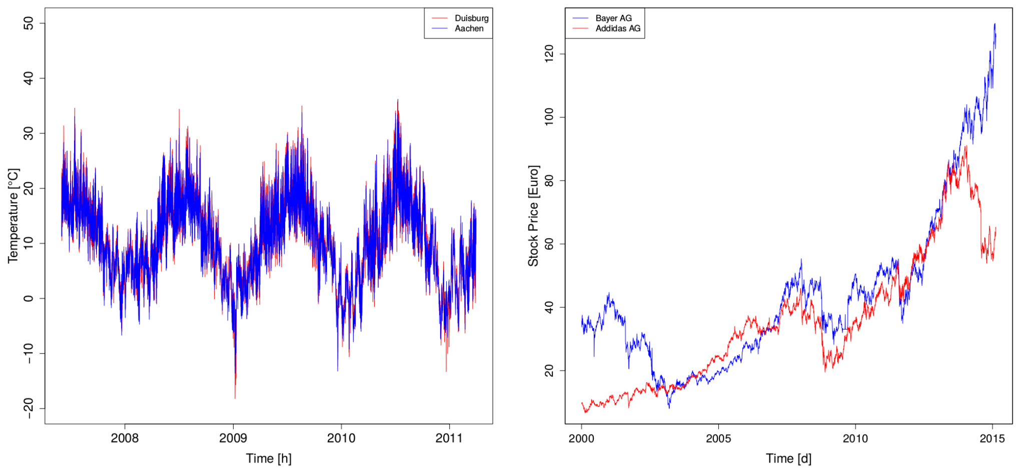


Figure 1.4: Left: Temperature time series on a hourly basis in Duisburg (red) and Aachen (blue) over a period from 1st of June 2007 till 31th of March 2011. Data is taken from http://www.dwd.de. Right: Stock price on a daily basis of the Bayer AG (blue) and the Addidas AG (red) stocks over a period from 3rd of January 2000 till 16th of February 2015. Data is taken from https://de.finance.yahoo.com.

by random numbers. However, they are not independently distributed, they show dependence. To quantify it Pearson introduced a correlation coefficient, the so called Pearson correlation coefficient. It measures the linear dependences between two time series X_1 and X_2. We will introduce it quantitatively in section 2.2.1. Depending on the time series, it is either a real, complex (or real quaternion) number with norm smaller or equal to one. For a real time series, a correlation coefficient of two time series X_1 and X_2 close to one means positive correlation such that both are most likely related by a linear function with positive slope. If it is close to minus one, both are most likely related by a linear function with negative slope. From a correlation coefficient of zero it follows that X_1 and X_2 are not linearly related. The Pearson correlation coefficient between the temperature time series in Duisburg and Aachen is illustrated in Fig. 1.5. Although the left plot in Fig. 1.4 suggest that the correlation coefficient should be close to one, we obtain large fluctuations, because we average over distinct windows of 24 hours only such that the seasonality does not play a role. For complex and real quaternion time series the interpretation of the correlation coefficient is similar.

If more than just two time series are observed or measured, we obtain several correlation coefficients which we order into the so called *correlation matrix*. In Fig. 1.3, we show the correlation matrix of eight different states of the S&P-500 index computed in Ref. [14].

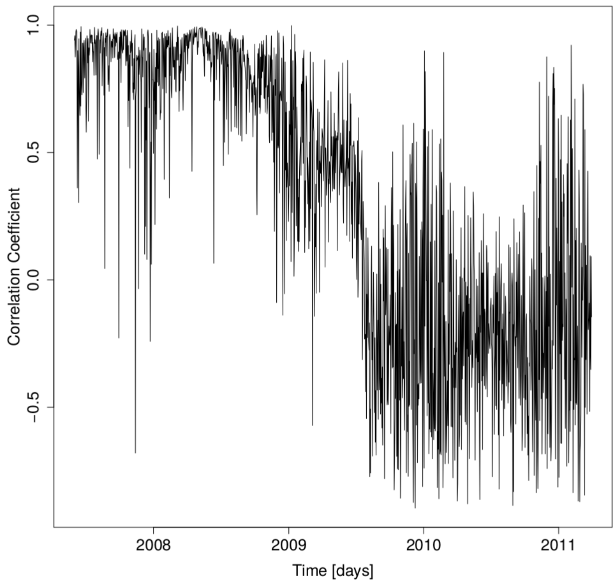


Figure 1.5: Pearson correlation coefficient between the temperature time series of Duisburg and Aachen obtained by averaging over distinct windows of 24 hours over a period from 1st of June 2007 till 31th of March 2011. Data is taken from http://www.dwd.de .

The correlation matrix and its eigenvalues play a central role in time series analysis and therefore in many applications of it. These include fields such as medicine [18], biology [19–21], geology [22, 23], chemistry [24–26], astronomy [27], finance market [28–30] and wireless communication [31–33]. Contrary to these fields where the correlations have a direct interpretation, it has as well a significant impact on the methodology in applications ranging over pattern recognition [34,35], machine learning [36], data classification [17], principal component analysis [16,37], detection of outliers [38] and network analysis [39], to name just a few.

Time series, observed or measured in experiments are subjected to statistical fluctuations. Accordingly, the correlation coefficient inherits this property and becomes itself a fluctuating quantity. Depending on the size of the sample, i.e. the number of time steps within each time series, it has strong fluctuations around its mean value. Notably, in the majority of examples, given above, the empirically estimated correlation matrix is taken as an approximately non-fluctuating quantity. The fluctuations are not considered at all. The question therefore arises:

What are the consequences of the fluctuations and how can we model them?

The latter question will be answered in the next section. Certain aspects of the former are considered in the remainder of this thesis.

1.2 Ensemble Approach

To study the consequences of the inherited fluctuations for the correlation matrix, we use an ensemble approach [37,40–44]. We replace the constituents of the correlations matrix, the empirical sample of time series, by an ensemble. It is fixed by the global symmetries of the observed data and a probability distribution. The former means we choose either a real, a complex or a real quaternion ensemble. The latter is fixed by the requirement that the ensemble has upon average the same correlation structure as the sample. It is encoded in a correlation matrix computed in an empirical sample and is given model input. Throughout this thesis we refer to it as empirical correlation matrix denoted by C.

In good agreement with empirical studies [23,28–30,45–48] we can model the fluctuations by an ensemble of Wishart correlation matrices, which we introduce later in more detail.

Following a brief historical introduction to random matrix theory and its application in the section 2.1, we will quantify the ensemble approach and introduce a random matrix model to study statistical properties of the correlation matrix in section 2.2.

CHAPTER 2

Theoretical Background

This chapter is devoted the theoretical background of this thesis. Except the extension of the generalized Hubbard-Stratonovich transformation to correlated Wishart models in section 2.5 which is the first result of this thesis, we only review aspects of random matrix theory, the correlated Wishart and Jacobi model, important statistical observables, basics of supermathematics, superbosonization and the generalized Hubbard-Stratonovich transformation known in the literature and important for later purpose.

In section 2.1, we give a brief overview of random matrix theory from the historical perspective and motivate upon the example of principal component analysis and canonical correlation analysis our approach. After this historical and motivating interlude, we quantify the ensemble approach and introduce the theoretical background of the correlated Wishart and Jacobi model. To study extreme eigenvalue statistics in the former and the spectral bulk statistics in both models, we introduce in section 2.3 statistical observables related to these issues. In section 2.4, we survey the mathematical backgrounds of supermathematics. These are used in sections 2.4.2 and 2.4.3 to review the method of superbosonization and the generalized Hubbard-Stratonovich transformation, respectively. As a first result of this thesis, we elaborate the extension of the generalized Hubbard-Stratonovich transformation to correlated Wishart models in section 2.5. We close this chapter with a summary in section 2.6.

2.1 Random Matrix Theory

Random matrix theory was originally introduced in the context of biostatistics by Wishart in Ref. [49]. He computed the probability distribution function of the correlation coefficient of real Gaussian distributed multivariates, generalizing earlier results for bivariates. This work was largely forgotten. Later, without prior knowledge of the results of Wishart, Wigner introduced Hermitian random matrix models in physics. Because for slow nuclear reactions not the locations but the statistics of excitations is important, his idea was to replace the Hamilton operator by an ensemble of random matrices with the same global symmetries [50]. This approach allows the study of the eigenvalue and eigenvector statistics of complex many-body quantum systems using random matrix theory.

Employing the idea of Wigner and comparing the distribution of the spacing of consecutive eigenvalues on the scale of mean level spacing for both experimental and random matrix data demonstrates the overwhelming agreement, see Fig. 2.1. It

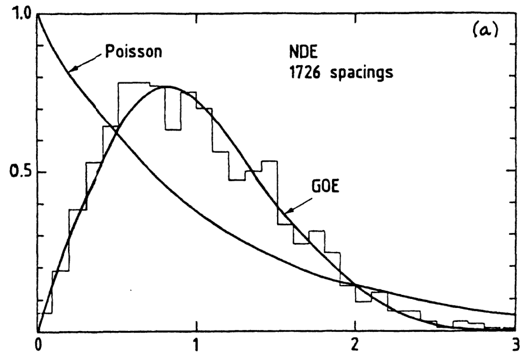


Figure 2.1: Comparision of Nearest-neighbor spacing distribution obtained from "Nuclear Data Ensemble" (histogram) to the Gaussian orthogonal ensemble (line) on the mean level spacing and the Poisson distribution (line). Taken from Ref. [51].

was argued by Wigner [52] and Dyson [53] that only three matrix ensembles have to be considered, the Gaussian orthogonal, the Gaussian unitary and the Gaussian symplectic ensemble. They are given by symmetric, Hermitian or real quaternion self-dual matrices, respectively, with Gaussian distributed entries.

Later, it was realized that apart from complex many-body quantum systems, random matrix theory applies to many other areas. These include fields like quantum chaos [54], quantum chromodynamics [55–58], two-dimensional quantum gravity [59], conformal field theory [60], integrable systems [61], condensate matter physics [62–65], quantum transport [66], representation theory [67–69], algebraic geometry [70], elastomechanics of irregularly shaped quartz crystal [71], quantum information theory [72] and the statistics of primes and zeros of the Riemann ζ-function [73, 74]. This list is being constantly extended by new applications, see Refs. [42, 43, 75] and references therein for reviews from a physical point of view.

In the majority of these applications, random matrix theory is used to model level fluctuations, because for these systems they turn out to be universal on the scale of the mean level spacing rather than system specific. Therefore, system typical information has to be "divided" out. By system typical we mean the absolute positions of levels encoded in the level density

$$\rho(E) = \sum_n \delta(E - E_n) \ , \tag{2.1}$$

where E_n are the individual levels, *i.e.* the eigenvalues of the Hamilton operator in the particular system or the locations of the primes or zeros of the Riemann ζ-function, *etc.* Figure 2.2 illustrates typical sequences of levels. To "divide" out

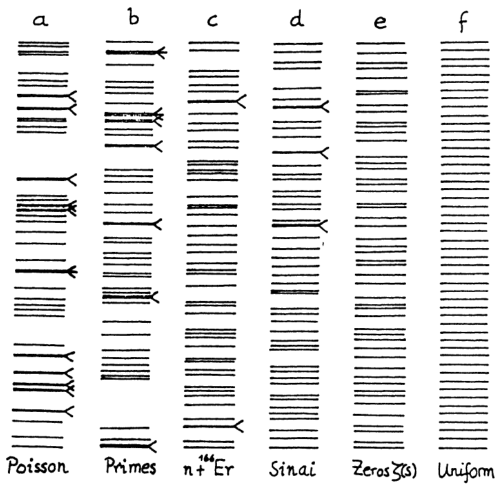


Figure 2.2: Five sequences of typical distributed levels and a sequence of uniformly distributed levels. (a) Non-correlated Poisson series, (b) sequence of prime numbers, (c) levels of slow neutron resonances for Erbium 166 nucleus, (d) energy levels of the Sinai billiard (e) zeros of the Riemann ζ-function on Re$s = 1/2$, (f) equally spaced levels. Taken from Ref. [76].

the mean level density, the unfolded eigenvalues

$$\xi_i = \int_{-\infty}^{\xi_i} \rho(E)\mathrm{d}E \tag{2.2}$$

are considered. By construction they have mean density one. Because, the fluctuations of these unfolded eigenvalues are conjectured to be universal [40, 42–44, 77], it does not matter whether they originate from a particular systems or a random matrix ensemble.

Independently, in mathematics the ideas of Wishart were refined in multivariate statistics [37] to study statistical properties of uncorrelated and correlated data, see Refs. [41, 78] and references therein. Rather than considering random matrix ensembles with infinite dimensions, in applications of multivariate statistics the focus is on finite size systems. In the past decades these ideas were applied in all fields of science to analyze and model correlations. Including fields like biology [79], numerical computation [80], chemistry [81], econophysics [29, 30], high dimensional inference [82], wireless communication [31, 32, 83], astrophysics [45], geology [47] and medicine [46] *etc.* In this thesis we concentrate on its applications to high-dimensional inference. Examples are given in the next section.

2.1.1 Applications to High-Dimensional Inference

The main field of applications of the correlated Wishart model is high-dimensional inference. The ensemble approach is used to study, improve and develop the methodology used in applied data analysis [37]. We illustrate its application with two examples.

To this end, we assume that we make n observations of a p-variate. This means, we have p sequences (time series) each of which has n time steps. It is supposed that these have mean zero, variance one and are ordered into the data matrix

$$M = \begin{bmatrix} M_1(1) & \cdots & M_1(n) \\ \vdots & \ddots & \vdots \\ M_p(1) & \cdots & M_p(n) \end{bmatrix} . \tag{2.3}$$

Principal component analysis is concerned with a dimensional reduction of the data [84]. It is an algorithm to determine principal directions or in other words the most significant subspace of $\mathbb{R}^p$ in the data. We introduce a new variate w with n steps by $w = v^T M$ such that $|v| = 1$. We require v to maximize the sample variance $\text{Var}(w) = 1/n \sum_{i=1}^p w_i^2 = v^T C v$, where C is the empirical correlation matrix

$$C = \frac{1}{n} M M^T . \tag{2.4}$$

The variance of w measures the projection in the sample of observations $M(i) = (M_1(i), \ldots, M_p(i))$ where $i = 1, \ldots, n$ onto v. If it is maximized, v is the principal direction in the p dimensional space. Because $v^T C v$ is a bilinear form, the vector v maximizing it is given by the eigenvector $\hat{v}_p$ to the largest eigenvalue Λ_p of C. The second significant direction is given by the eigenvector $\hat{v}_{p-1}$ to the second largest eigenvalue Λ_{p-1} of C and so on. Furthermore, the sample variance of w_j with $v = \hat{v}_j$ an eigenvector of C is given by

$$\text{Var}(w_j) = v_j^T C v_j = \Lambda_j. \tag{2.5}$$

According to $\Lambda_i \leq \Lambda_{i-1}$ for all $i = 2, \ldots, p$, we find that the variance of the different components decreases $\text{Var}(w_i) \leq \text{Var}(w_{i-1})$. To distinguish the significant from the non-significant directions, we compare their associated variances. It turns out that the percentage of information carried by the direction v_j with $j = 1, \ldots, p$ is given by

$$\frac{\text{Var}(w_j)}{\sum_{i=1}^p \text{Var}(w_i)} = \frac{\Lambda_j}{\sum_{i=1}^p \Lambda_i} = \Lambda_j , \tag{2.6}$$

where we make use of $\sum_{i=1}^p \Lambda_i = \text{tr}\, C = 1$, because the observations are normalized to variance one. Thus, the reduced data lies in the subspace of $\mathbb{R}^p$ spanned by those eigenvectors $\hat{v}_i$ for which Λ_i is significantly larger the gross of the eigenvalues. This leads to two elementary questions: *What are the fluctuations of the significant principal components? How do they depend on the correlation structure in the data and on the size of the sample?*

To answer these questions, we need more information about the data and make an ensemble approach, which means we replace the sample with an ensemble, see section 1.2. This leads to the real correlated Wishart model, where the data matrices M are replaced by model data matrices W of the same size. In this model, we are able to study the statistics of the large eigenvalues of $WW^\dagger$, which corresponds to the principal components in the ensemble.

Besides the large eigenvalues the smallest eigenvalue and the entire spectrum of the sample correlation matrix and therefore of the Wishart model correlation matrix are important for certain aspects in multivariate statistics. For instance, the smallest eigenvalue gives the leading contribution for the *threshold estimate* in linear discriminant analysis [17]. It is most sensitive to *noise* [16]. It is crucial for the identification of *single statistical outliers* [38]. In numerical studies involving large random matrices, the *condition number* is used, which depends on the smallest eigenvalue [80, 85].

Contrary to principal component analysis, in canonical correlation analysis two different observations are compared. We assume that we observed two sets of time series, normalized to zero mean and unit variance and ordered in the $p \times n$ and $q \times n$ dimensional data matrices M_1 and M_2, respectively. The canonical correlation of the samples is given by the Pearson correlation coefficient between a linear combination of the observation M_1 and the observation M_2 that is most correlated,

$$r = \max_{u,v} \mathrm{Corr}(u^T M_1, v^T M_2) \ , \tag{2.7}$$

where $u, v \in \mathbb{R}^p$ with $|u| = |v| = 1$. Here $\mathrm{Corr}(X, Y)$ is the Pearson correlation coefficient in the sample between the time series X and Y. Employing the ensemble approach and replacing the samples by ensembles, we obtain two independent Wishart model correlation matrices $FF^\dagger$ and $BB^\dagger$. From the analysis of Ref. [41], canonical correlation analysis results in the analysis of the roots r_i $i = 1, \ldots, p$ of

$$\det\left(r_i(FF^\dagger + BB^\dagger) - FF^\dagger\right) = 0 \ . \tag{2.8}$$

where F and B are real $p \times n_1$ and $p \times n_2$ matrices with $n_1 = q$, $n_2 = n - q$ and the assumption $n_1, n_2 \geq p$. In the mathematical literature [37, 41, 82] this ensemble is known as the double Wishart model. Studying the roots in Eq. (2.8) is similar to studying the eigenvalues of the correlated Jacobi ensemble [44]. It consists of Hermitian matrices

$$\mathcal{H} = \frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger} \ , \tag{2.9}$$

where F and B are as introduced above. Besides the canonical correlation analysis, the eigenvalue statistics of $\mathcal{H}$ is crucial for multivariate analysis of variances, multivariate regression analysis and the test of equality of correlation matrices [37, 41, 86, 87].

We want to emphasize that in general the assumption of data normalized to unit variance is dropped when principal component or canonical correlation analysis are introduced. This corresponds to replacing the correlation matrix C above by the covariance matrix Σ. They are both related by $\Sigma = \sigma^{-1} C \sigma^{-1}$, where $\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$ with $\sigma_i = \sqrt{\mathrm{Var}(X_i)}$ and X_i the sample time series.

2.2 Correlated Wishart and Jacobi Model

This section reviews the correlated Wishart and Jacobi model. In section 2.2.1 we put the ensemble approach outlined in section 1.2 on mathematical grounds and thereby introduce the correlated Wishart model. It has two important limiting cases, which we briefly discuss in an interlude in section 2.2.2. Building on section 2.2.1, we introduce in section 2.2.3 the correlated Jacobi ensemble consisting of two independent correlated Wishart models. To study the eigenvalue statistics in the correlated Wishart, we compute in section 2.2.4 its joint eigenvalue distribution function.

2.2.1 Introducing the Correlated Wishart Model

After a qualitative introduction to the ensemble approach in section 1.2, we introduce it here in a precise mathematical manner.

We assume that we have a set of p time series X_i, each with n time steps. The entries $X_i(t)$ for $i = 1, \ldots, p$ and $t = 1, \ldots, n$ are either real, complex or real quaternion self-dual. Although the latter is less important for applications in time series analysis, we include it in a unifying way. As explained in section 1.1, linear correlations between different time series are measured by Pearson's correlation coefficient [15]

$$\rho_{ij} = \left\langle \frac{\left(X_i - \langle X_i \rangle_{\mathrm{s}}\right)\left(X_j - \langle X_j \rangle_{\mathrm{s}}\right)^*}{\sqrt{\left\langle X_i^2 \right\rangle_{\mathrm{s}} - \langle X_i \rangle_{\mathrm{s}}^2}} \right\rangle_{\mathrm{s}} , \tag{2.10}$$

where $*$ is the complex conjugation, i and j are fixed and $\langle \cdot \rangle_{\mathrm{s}}$ is the sample average over time,

$$\langle f(X) \rangle_{\mathrm{s}} = \frac{1}{n} \sum_{t=1}^{n} f(X(t)) . \tag{2.11}$$

For real time series it varies between -1 and 1. For complex and real quaternionic time series, ρ_{ij} is complex or real quaternion number as well such that its norm is less than or equal to one. In this case the interpretation is more delicate.

If we normalize the time series to zero mean and unit variance,

$$M_i(t) = \frac{X_i(t) - \langle X_i \rangle_{\mathrm{s}}}{\sqrt{\left\langle X_i^2 \right\rangle_{\mathrm{s}} - \langle X_i \rangle_{\mathrm{s}}^2}} , \tag{2.12}$$

and order them into a $p \times n$ dimensional matrix, the so-called *data matrix*

$$M = \begin{bmatrix} M_1(1) & \cdots & M_1(n) \\ \vdots & \ddots & \cdots \\ M_p(1) & \cdots & M_p(n) \end{bmatrix} , \tag{2.13}$$

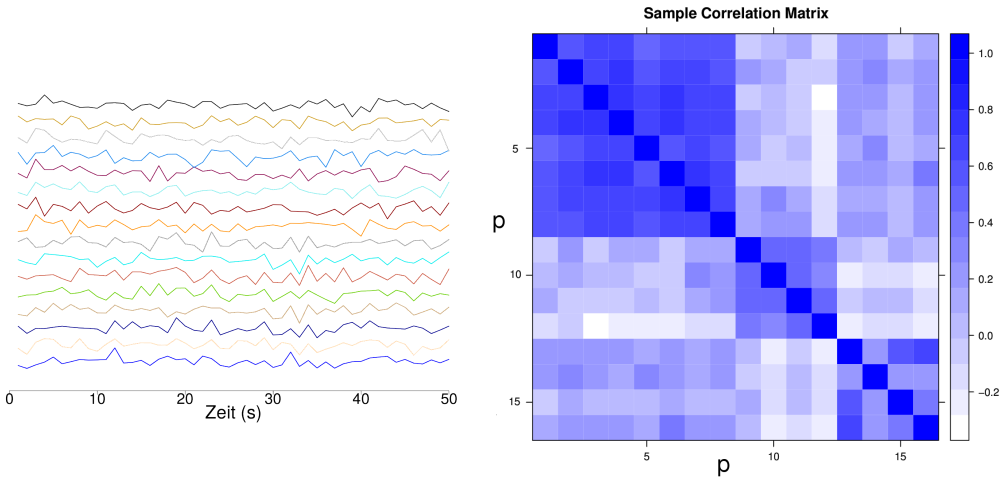


Figure 2.3: Left: A numerically generated sample of 16 time series each with 50 time steps. Right : The correlation matrix corresponding to the set of time series. It shows that there is indeed a non-trivial correlation structure in this sample.

we can introduce the sample correlation matrix as

$$C = \frac{1}{n} M M^\dagger \ . \tag{2.14}$$

The matrix C is by definition a $p \times p$ positive semi-definite real symmetric, Hermitian, real quaternion self-dual matrix for $\beta = 1, 2, 4$, respectively. Its entries C_{ij} are the Pearson correlation coefficient between the two time series X_i and X_j. Throughout this thesis, we refer to M and C as the empirical data and correlation matrix, respectively. To illustrate the procedure discussed above, we show in the left plot of Fig. 2.3 an example of 16 time series, each with 50 time steps. For this sample we obtain the correlation matrix shown in the right of Fig. 2.3. Further examples of correlation matrices are given in Fig. 1.3.

As explained in section 1.1, the correlation matrix is of tremendous importance in time series analysis. To model its fluctuations around the sample mean, random matrix theory serves as a natural candidate, because it can be used to model generic and system specific features of the correlations.

We use an ensemble approach, see also section 1.2, and replace the empirical data matrix (2.13) by a rectangular model data matrix W such that on average

$$\frac{1}{n} \left\langle W W^\dagger \right\rangle = C \ . \tag{2.15}$$

In this way we introduce an ensemble of model correlation matrices $WW^\dagger/n$ fluctuating around an empirical correlation matrix C, where C is thought of as a mean correlation matrix determined in a sample of time series such that it has full rank and "enough" information about the correlation structure in this sample. It results from observations or experiments and is given model input.

To fix an ensemble of model data matrices W and therefore of model correlation matrices $WW^\dagger/n$, we choose W to be a rectangular $p \times n$ matrix with entries $W_{ij} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ for C to be a real symmetric, Hermitian, real quaternion self-dual matrix, denoted by $\beta = 1, 2, 4$, respectively. The number of rows p and columns n of W correspond to the number of model time series, respectively, their number of time steps. We restrict the case to where $n \geq p$ such that W has rank p. In good agreement with empirical observations [23, 28–31, 44–48, 82, 88], we can take the entries W_{ij} of W to be normal distributed with variance C_{ji} so that [37, 41, 44]

$$P(W|C) = \frac{\exp\left(-\frac{\beta}{2}\mathrm{tr}WW^\dagger C^{-1}\right)}{(\beta/2\pi)^{np\beta/2}\mathrm{det}^{np/\gamma_1}C} , \tag{2.16}$$

where we introduce $\gamma_1 = 1$ for $\beta = 2, 4$ and $\gamma_1 = 2$ for $\beta = 1$ and for later purpose $\gamma_2 = \gamma_1\beta/2$ and $\tilde{\gamma} = \gamma_1\gamma_2$. The ensemble just constructed is the real, complex or real quaternion *correlated Wishart model* and $WW^\dagger/n$ the *Wishart model correlation matrix*.

On the space $\mathrm{Mat}_{p\times n}(\mathbb{K})$ of $p \times n$-dimensional rectangular matrices W with entries in $\mathbb{K} = \mathbb{R}, \mathbb{C}$, respectively, $\mathbb{H}$, we take the flat measure $\mathrm{d}[W]$. Throughout this thesis, $\mathrm{d}[\cdot]$ denotes the flat measure, which means the product of all independent differentials.

2.2.2 Hard and Soft-Edge Limit

Since Wishart model correlation matrices are introduced as dyadic product of rectangular model data matrices, different limiting regimes have to be distinguished. The rectangular data matrices have p rows and n columns. Thus, if we want to compute large n, p limits of statistical quantities, we have to fix how n and p behave mutually when they tend to infinity. It turns out that two regimes are of particular interest.

The first is the hard edge limit, where the rectangularity $\nu = n - p$ is kept fixed when n, p tend to infinity. Its name originates from the observation that in the case of the uncorrelated Wishart model, the smallest eigenvalue approaches zero. In Ref. [89] it was shown that it tends to zero with a rate of $\mathcal{O}(p^{-1})$. Thus, if we rescale the eigenvalues by p, we can study the statistics of the low lying eigenvalues, which feel the presence of a hard wall in the spectrum at zero. This limit is particularly import for the random matrix approach of quantum chromodynamics, to study the statistics of the low-lying eigenvalues of the Dirac operator [55, 58, 90].

The second limit is the soft edge scaling. If $\gamma^2 = p/n < 1$ is fixed for n, p tending to infinity, the low lying eigenvalues in the spectra do not feel the presence of the wall at zero. In this case the largest and the smallest eigenvalue have the same statistics [91]. This limit is particularly important for applications in high dimensional inference, where the number of time steps n is much larger than the number of measurements p but both are still large, see also Refs. [82, 92, 93] and references therein.

2.2.3 Introducing the Correlated Jacobi Ensemble

The correlated Jacobi ensemble, also referred to as double Wishart model arises if two observations are involved in an analysis, see section 2.1.1. In this case the two samples are independently replaced by an ensemble of Wishart correlation matrices. These can be real, complex or real quaternion depending on the data.

If F and B are either real, complex or real-quaternion correlated $p \times n_1$, respectively, $p \times n_2$ dimensional data matrices with $n_1, n_2 \geq p$, the correlated Jacobi matrix is introduced as

$$\mathcal{H} = \frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger} \ . \tag{2.17}$$

Its eigenvalues are real and its spectrum is by construction bound to be spec$(\mathcal{H}) \subset [-1, 1]$. Both F and B are distributed with respect to Eq. (2.16), but with different empirical correlation matrices C_F and C_B, respectively. The distribution of $\mathcal{H}$ is at least formally given as the average over the F and B ensemble,

$$P(\mathcal{H}) = \int \mathrm{d}[B, F] \ P(F|C_F)P(B|C_B)\delta\left(\mathcal{H} - \frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger}\right) \ . \tag{2.18}$$

On account of the structure of $\mathcal{H}$, the computation of the distribution function $P(\mathcal{H})$ is even for the case of the complex Wishart ensemble a non-trivial task. If it happens that $C_B = C_F = C$, the distribution $P(\mathcal{H})$ is independent of C. Diagonalization of the resulting Hermitian matrix model leads to an eigenvalue ensemble distributed with respect to the Jacobi weight [40, 44]. This case was intensively studied in the mathematical literature [41]. To the best of our knowledge, for the case of $C_F \neq C_B$ only a few estimates are known [37, 94].

2.2.4 Joint Eigenvalue Distribution Function

In the section 2.2.1, we fix $n \geq p$ such that $WW^\dagger$ has full rank and $W^\dagger W$ has $n - p$ zero eigenvalues. To obtain the joint eigenvalue distribution function, we diagonalize $WW^\dagger = U(\mathbf{1}_{\gamma_2} \otimes X)U^\dagger$, where $\mathbf{1}_N$ is the unit matrix in N dimensions. Here $X = \mathrm{diag}\,(x_1, \ldots, x_p)$ is the matrix of distinct eigenvalues of $WW^\dagger$ and U is an element of the group of $p \times p$ orthogonal $\mathrm{O}(p)$, unitary $\mathrm{U}(p)$ or unitary symplectic $\mathrm{USp}(2p)$ matrices for $\beta = 1, 2, 4$, respectively. We introduce G_p as a short-hand notation for either $\mathrm{O}(p)$, $\mathrm{U}(p)$ or $\mathrm{USp}(2p)$ whenever $\beta = 1, 2$ or 4, respectively. Diagonalization of the integration measure leads to a decomposition of the volume element $\mathrm{d}[W]$ [44]

$$\mathrm{d}[W] \ \sim |\Delta_p(X)|^\beta \prod_{i=1}^{p} x_i^{\beta(n-p+1-2/\beta)/2} \mathrm{d}[X] \ \mathrm{d}\mu(U) \ , \tag{2.19}$$

where $\Delta_p(X) = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant and $\mathrm{d}\mu(U)$ is the Haar measure on G_p. Averaging the probability distribution (2.16) with respect to

the Haar measure over G_p, yields the joint eigenvalue distribution function,

$$P(X|C) \sim |\Delta_p(X)|^\beta \prod_{i=1}^{p} x_i^{\beta(n-p+1-2/\beta)/2} \Phi_\beta(X, C^{-1}) \ . \tag{2.20}$$

If $C = \mathbf{1}_p$, it turns out that $\Phi_\beta(X, C^{-1}) = \exp(-\beta/2\mathrm{tr}X)$. The resulting distribution $P(X|\mathbf{1}_p)$ coincides with the Laguerre ensemble and is therefore interchangeably referred to as uncorrelated Wishart or Wishart-Laguerre ensemble. If $C \neq \mathbf{1}_p$, the function Φ_β is a highly non-trivial group integral and given by

$$\Phi_\beta(X, C^{-1}) = \int d\mu(U) \exp\left(-\frac{\beta}{2}\mathrm{tr}UXU^\dagger C^{-1}\right) \ . \tag{2.21}$$

It is known as *Harish-Chandra-Itzykson-Zuber* integral [95, 96] for $\beta = 2$ or as the *orthogonal*, respectively, *unitary-symplectic Itzykson-Zuber* integral for $\beta = 1, 4$. For $\beta = 2$ an analytic expression exists, because X as well as C^{-1} are Hermitian matrices and therefore are elements of the Lie algebra of $\mathrm{U}(p)$. In this case, the saddle point approximation of Eq. (2.21) becomes exact [97] such that analytic calculations are possible leading to a closed-form expression [95, 96]. For $\beta = 1, 4$ neither X nor C^{-1} are elements in the Lie algebra of $\mathrm{O}(p)$ if $\beta = 1$ or $\mathrm{USp}(2p)$ if $\beta = 4$. Therefore no results are known in this cases. Only in the special case of the real and real quaternion spiked Wishart model the corresponding Itzykson-Zuber integral is known. Explicit results have been given in Ref. [98] and Ref. [99]. In any other case there it remains interesting to find an analytic closed-form expression for these integrals.

The Haar measure $d\mu(U)$ on G_p has the unique property to be invariant under left and right action of $V \in \mathrm{G}_p$. Thus, the Itzykson-Zuber integral depends only on the eigenvalues of $C = V\hat{\Lambda}V^\dagger$ with $\hat{\Lambda} = \mathbf{1}_{\gamma_2} \otimes \Lambda$. We order the always non-negative, distinct eigenvalues of C into the diagonal matrix $\Lambda = \mathrm{diag}\,(\Lambda_1, \ldots, \Lambda_p)$, referred to as *empirical eigenvalue matrix*.

In order to study the eigenvalue statistics in the Wishart model further, mathematicians introduced the so called Jack polynomials. They constitute a family of multivariate orthogonal polynomials depending on one parameter $\alpha = 2/\beta$ [100]. For $\beta = 1, 2, 4$ these are given by the zonal, the Schur and the quaternion zonal polynomials, respectively. We follow Ref. [44] and introduce them as homogeneous polynomial eigenfunctions $C_\lambda^\alpha(X)$ of the Laplace-Beltram differential operator

$$D_p C_\lambda^\alpha(X) = \left(\rho_\lambda^\alpha + k(p-1)\right) C_\lambda^\alpha(X) \tag{2.22}$$

where

$$D_p = \sum_{i=1}^{p} x_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1\leq i<j\leq p} \frac{x_i^2}{x_i - x_j}\frac{\partial}{\partial x_i} \ , \tag{2.23}$$

$\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ is a partition of k of length $l(\lambda)$, p is the number of variables x_i and $\rho_\lambda^\alpha = \sum_{i=1} \lambda_i \left(\lambda_i - 1 - 2(i-1)/\alpha\right)$. We focus on the case where $\beta = 1$ and

$C^2_\lambda(X)$ are given by the zonal polynomials. These have the property that [41]

$$\mathrm{tr}^m X = \sum_{\lambda\vdash m} C^2_\lambda(X) \ , \tag{2.24}$$

where $\lambda \vdash m$ means that λ is a partition of m. Employing the identity above we write the exponent in Eq. (2.21) as

$$\exp\left(-\frac{\beta}{2}\mathrm{tr}UXU^\dagger\Lambda^{-1}\right) = \sum_{i=0}^{\infty}\frac{(-1)^i\mathrm{tr}^i\left(UXU^\dagger\Lambda^{-1}\right)}{2^i i!} = \sum_{i=0}^{\infty}\sum_{\lambda\vdash i}\frac{(-1)^i C^2_\lambda\left(UXU^\dagger\Lambda^{-1}\right)}{2^i i!} \ . \tag{2.25}$$

If we insert this expansion into the Itzykson-Zuber integral, exchange the summation and integration, we are left with an integral of the zonal polynomials with respect to the Haar measure on the orthogonal group. They are given by [41]

$$\int \mathrm{d}\mu(U) C^2_\lambda\left(UXU^\dagger\Lambda^{-1}\right) = \frac{C^2_\lambda(X)C^2_\lambda(\Lambda^{-1})}{C^2_\lambda(\mathbf{1}_p)} \tag{2.26}$$

such that the integral (2.21) becomes an infinite sum running over all permutation that reads

$$\Phi_1(X,\Lambda^{-1}) = \sum_{i=0}^{\infty}\sum_{\lambda\vdash i}\frac{(-1)^i C^2_\lambda(X)C^2_\lambda(\Lambda^{-1})}{2^i C^2_\lambda(\mathbf{1}_p) i!} \ . \tag{2.27}$$

The Itzykson-Zuber integral belongs to a large class of invariant functions which all have a representation in terms of an infinite sum as shown in Eq. (2.27). These are referred to as hypergeometric functions of matrix argument, see Refs. [41, 44, 101, 102]. The drawback of this representation is that no analytic closed-form expression for the polynomials is known, neither for the zonal C^2_λ nor for the Jack polynomials C^α_λ . They have to be constructed iteratively on a computer, see Ref. [103] for recent results.

2.3 Statistical Quantities

In this section, we review statistical quantities which are analyzed in the upcoming sections. All observables considered in this thesis depend on the eigenvalues of the Wishart matrix $WW^\dagger$ only and have a representation in terms of an eigenvalue integral. We construct for each of which a matrix model average for later use.

We start in section 2.3.1 with the level density and the k-point correlation function. We then turn in section 2.3.2 to the statistics of the largest eigenvalue and complement it with the statistics of the smallest eigenvalue in section 2.3.3.

2.3.1 Level Density and k-Point Function

The ensembles considered here consist of real symmetric, Hermitian or real quaternion self-dual $p \times p$ matrices H, drawn from a distribution function $P(H)$. For the correlated Wishart and Jacobi model we set $H = WW^\dagger$ and $H = (FF^\dagger - BB^\dagger)(FF^\dagger + BB^\dagger)^{-1}$ and take the distribution to be given by Eq. (2.16) and Eq. (2.18), respectively. If we diagonalize $H = U(X \otimes \mathbf{1}_{\gamma_2})U^\dagger$, where $U \in \mathrm{G}_p$ and $X = \mathrm{diag}(x_1, \ldots, x_p)$ the matrix distinct real eigenvalues of H, the level density is given by [40, 44]

$$\begin{aligned} S(x) &= \left\langle \frac{1}{p} \sum_{i=1}^{p} \delta(x_i - x) \right\rangle \\ &= \frac{1}{p} \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_p \sum_{i=1}^{p} \delta(x_i - x) P(x_1, \ldots, x_p) \ . \end{aligned} \tag{2.28}$$

It is normalized such that $\int_{-\infty}^{\infty} S(x)\mathrm{d}x = 1$. In expression (2.28) $P(x_1, \ldots, x_p)$ is the joint probability distribution function. It can always be written as an average of the matrix distribution $P(H)$ over the diagonalizing group G_p ,

$$P(x_1, \ldots, x_p) \sim |\Delta_p(X)|^\beta \int \mathrm{d}\mu(U) P(UXU^\dagger) \ . \tag{2.29}$$

The Vandermonde determinant results as Jacobian from the diagonalization of the measure. For the Wishart model $P(X)$ is worked out in section 2.2.4 and is given by Eq. (2.20). To get rid of the δ-function, we can use the following identity

$$\mp \pi \delta(x) = \mathrm{Im} \lim_{\epsilon \to 0} \frac{1}{x \pm \imath\epsilon} \ . \tag{2.30}$$

If we insert this into Eq. (2.28), it turns out that the density is given by the imaginary part of the averaged trace of the resolvent

$$S(x) = \frac{1}{\pi p} \mathrm{Im} \lim_{\epsilon \to 0} \int \mathrm{d}[X] \ \mathrm{tr} \frac{\mathbf{1}_p}{X - x^+ \mathbf{1}_p} P(X) \ , \tag{2.31}$$

where $x^\pm = x \pm \imath\epsilon$. The trace in Eq. (2.31) is invariant under $X \mapsto VXV^\dagger$, with $V \in \mathrm{G}_p$. Thus, because the X integral is over the whole spectrum of H, we express the integrals in Eq. (2.31) as a full matrix model average,

$$S(x) = \frac{1}{\pi p} \mathrm{Im} \lim_{\epsilon \to 0} \int \mathrm{d}[H] \ \mathrm{tr} \frac{\mathbf{1}_p}{H - x^+ \mathbf{1}_p} P(H) \ . \tag{2.32}$$

For later purpose, we want to write this matrix integral as an average over a ratio of determinants using

$$\frac{\partial}{\partial j}\bigg|_{j=0} \frac{\det (A + j\mathbf{1}_p)}{\det A} = \frac{1}{2} \frac{\partial}{\partial j}\bigg|_{j=0} \frac{\det (A + j\mathbf{1}_p)}{\det (A - j\mathbf{1}_p)} = \mathrm{tr} A^{-1} \tag{2.33}$$

Substituting Eq. (2.33) into Eq. (2.32), we find the desired matrix model for the level density. It is given in terms of an average with respect to $P(H)$ of a ratio of determinants,

$$S(x) = \frac{1}{\pi p} \frac{\partial}{\partial j}\bigg|_{j=0} \mathrm{Im} \lim_{\epsilon \to 0} \int \mathrm{d}[H] \ \frac{\det (H - x^+ \mathbf{1}_p + j\mathbf{1}_p)}{\det (H - x^+ \mathbf{1}_p)} P(H) \ . \tag{2.34}$$

The k-point correlation function is the joint probability distribution function of k out p eigenvalues of a Hermitian matrix H. It derives from the joint probability distribution function by integrating $P(X)$ over $p-k$ out of p eigenvalues. If we apply a similar analysis as in the case of the density, we obtain

$$\begin{aligned} R_k(x_1,\dots,x_k) &= \frac{(p-k)!}{(4\pi\imath)^k p!} \sum_{L\in\{\pm1\}^k} \prod_{b=1}^{k} L_b \partial_{j_b} \\ &\times \int \mathrm{d}[H]\ P(H) \prod_{a=1}^{k} \left. \frac{\det\left(H-(x_a+\imath\varepsilon L_a - j_a)\mathbf{1}_p\right)}{\det\left(H-(x_a+\imath\varepsilon L_a + j_a)\mathbf{1}_p\right)} \right|_{j=0}, \end{aligned} \tag{2.35}$$

where $L = \mathrm{diag}(L_1,\dots,L_k)$ and $j = \mathrm{diag}(j_1,\dots,j_k)$. It is normalized such that $\int_{-\infty}^{\infty} \mathrm{d}x_1 \cdots \mathrm{d}x_k R_k(x_1,\dots,x_k) = 1$. The j_a with $a = 1,\dots,k$ are source variables generating a trace of the inverse resolvent such that the sum over the sign L_a of the imaginary increment $\imath\varepsilon$ in Eq. (2.35) projects out only the imaginary part of these traces. For $k=1$ the k-point function $R_k(x_1,\dots,x_k)$ reduces to the level density $S(x)$.

The average of ratios of characteristic polynomials in the second row of Eq. (2.35), is known as the *generating function* for the k-point function. These averages are well studied objects for random matrix ensembles with invariant probability distribution functions, see Refs. [40,42–44,104–114]. However, because of the appearance of the Itzykson-Zuber integral in Eq. (2.20), in the correlated Wishart model it can not be studied using these results. The only possibility to gain new insights for the observables discussed in the current section is the method of supersymmetry, see section 2.5.

2.3.2 Largest Eigenvalue Distribution

In contrast to the level density and the k-point function which are mainly concerned with bulk statistics, *i.e.* where the gross of eigenvalues are, the largest eigenvalue distribution is in some sense a much more "local" quantity. Local means that significant contributions to the distribution come from the upper edge of the spectrum.

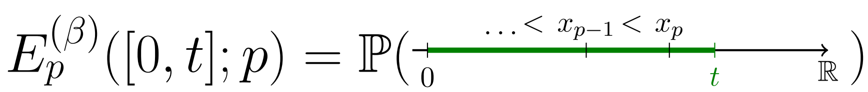


Figure 2.4: The gap probability to find all p eigenvalues of a matrix within the interval $[0,t]$.

To study the statistics of the largest eigenvalue in the correlated Wishart model, we utilize the gap probability to find all eigenvalues of $WW^\dagger$ within $[0,t]$. This gap probability is shown in Fig. 2.4, where $E_p^{(\beta)}([a,b];m)$ is the probability of finding m out of p eigenvalues in the interval $[a,b]$.

We assume that we know the distribution of the largest eigenvalue $\mathcal{P}^{(\beta)}_{\mathsf{max}}(t)$. The gap probability to find all eigenvalues below a threshold t is the cumulative density function of the largest eigenvalue,

$$E_p^{(\beta)}([0,t];p) = \int_0^t \mathrm{d}t' \mathcal{P}^{(\beta)}_{\mathsf{max}}(t') \quad \Leftrightarrow \quad \mathcal{P}^{(\beta)}_{\mathsf{max}}(t) = \frac{\mathrm{d}}{\mathrm{d}t} E_p^{(\beta)}([0,t];p) \ . \tag{2.36}$$

Thus, if we knew the gap probability, we knew the distribution of the largest eigenvalue. The gap probability is known in terms of the joint probability distribution function (2.20),

$$E_p^{(\beta)}([0,t];p) = \int\limits_0^t \mathrm{d}x_1 \cdots \int\limits_0^t \mathrm{d}x_p P(X|\hat{\Lambda}^{-1}) \ . \tag{2.37}$$

In contrast to all other statistical quantities considered in this thesis, the gap probability above is given by a compact integral. This property makes it a challenging to find an underlying full Wishart model dual to Eq. (2.37) without using a Heaviside Θ-function. The disadvantage of the Θ-functions is that they are difficult to handle analytically.

However, to obtain an matrix model representation of the gap probability (2.37), we extend the integral (2.37) to the entire spectrum of $WW^\dagger$, and are left with

$$E_p^{(\beta)}([0,t];p) = \int\limits_0^\infty \mathrm{d}x_1 \cdots \int\limits_0^\infty \mathrm{d}x_p P(X|\hat{\Lambda}^{-1}) \prod_{i=1}^p \Theta(t - x_i), \tag{2.38}$$

where $\Theta(x)$ is the Heaviside Θ-function and is one if $x > 0$ and zero otherwise. We use the same symbol for the Θ-function of matrix argument, which we introduce as $\Theta(A) = 1$ if A is positive definite and zero otherwise. Because the integral (2.38) is over the whole spectrum of $WW^\dagger$ and the Θ-function is invariant under base transformations, *i.e.* $\Theta(UAU^\dagger) = \Theta(A)$ with $U \in \mathrm{G}_p$, we find

$$\prod_{i=1}^p \Theta(t - x_i) = \Theta\left(t\mathbf{1}_p - X\right) = \Theta\left(t\mathbf{1}_{\gamma_2 p} - WW^\dagger\right) \ . \tag{2.39}$$

We can apply the steps leading to Eq. (2.20) in the backward direction and arrive at

$$E_p^{(\beta)}([0,t];p) = \int \mathrm{d}[W] \ P(W|\hat{\Lambda}^{-1}) \Theta(t\mathbf{1}_{\gamma_2 p} - WW^\dagger) \ . \tag{2.40}$$

Here W is a real, complex or real quaternion $p \times n$ matrix. As we will see, the full matrix model average (2.40) serves as a good starting point to study the gap probability (2.37).

2.3.3 Smallest Eigenvalue Distribution

Although we are concerned with the smallest eigenvalue statistics in the uncorrelated Wishart model in section 3, this section exclusively deals with its statistics in the correlated Wishart model. The former is a special case of the latter and we postpone its discussion to the particular sections.

The distribution of the smallest eigenvalue is the probability that all eigenvalues except one lie in $[s,\infty)$ and one in $[s-\mathrm{d}s, s]$. In terms of the joint probability distribution function (2.20) it reads

$$\mathcal{P}_{\text{min}}^{(\beta)}(s) = \sum_{i=1}^{p} \int\limits_s^\infty \mathrm{d}x_1 \cdots \int\limits_s^\infty \mathrm{d}x_{i-1} \int\limits_s^\infty \mathrm{d}x_{i+1} \cdots \int\limits_s^\infty \mathrm{d}x_p P(X|\hat{\Lambda}) \Bigg|_{x_i=s} , \tag{2.41}$$

where the sum over all p eigenvalues symmetrizes the quantity. For the uncorrelated Wishart model the sum in Eq. (2.41) reduces to a factor of p times a $p-1$ fold integral. This is due to the invariance of the integrand under the permutation of integration variables.

To study the smallest eigenvalue distribution, we construct for later purpose a Wishart model average. As we are integrating in Eq. (2.41) over $p-1$ of the p eigenvalues of $WW^\dagger$ only, we can not simply find a general matrix model representation dual to Eq. (2.41). But, the particular form of Eq. (2.41) suggests instead of Eq. (2.41) to analyze

$$\mathcal{P}_{\text{min}}^{(\beta)}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} \int\limits_s^\infty \mathrm{d}x_1 \cdots \int\limits_s^\infty \mathrm{d}x_p P(X|\hat{\Lambda}) = -\frac{\mathrm{d}}{\mathrm{d}s} E_p^{(\beta)}([0,s];0) \ . \tag{2.42}$$

Thus, the smallest eigenvalue distribution is proportional to a derivative of the gap probability to find none of the p eigenvalues within the interval $[0,s]$. This gap probability is schematically explained in Fig. 2.5.

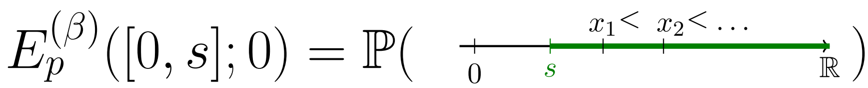


Figure 2.5: The gap probability to find no eigenvalue of a Wishart matrix within the interval $[0,s]$.

Analogous to the cumulative density function of the largest eigenvalue (2.37), is $E_p^{(\beta)}([0,s];0)$ given in terms of an eigenvalue integral. Thus, we find a dual full matrix model representation if we use the Heaviside Θ-function,

$$E_p^{(\beta)}([0,s];0) = \int\limits_0^\infty \mathrm{d}x_1 \cdots \int\limits_0^\infty \mathrm{d}x_p P(X|\hat{\Lambda}) \Theta(X - s\mathbf{1}_p) \ . \tag{2.43}$$

Applying the analysis done in section 2.2.4 backwards and employing the invariance of the Heaviside Θ-function with respect to base changes, see Eq. (2.39), we arrive at

$$E_p^{(\beta)}([0,s];0) = \int \mathrm{d}[W]\ P(W|\hat{\Lambda})\Theta(WW^\dagger - s\mathbf{1}_{\gamma_2 p}), \tag{2.44}$$

where W is a $p \times n$ matrix with real, complex or real quaternion entries.

2.4 Supersymmetry and Supermathematics

In this section we introduce the mathematical details of the supersymmetry method. We summarize in section 2.4.1 the linear algebra of supervectorspaces and supermatrices and introduce the integrals over superdomains. In section 2.4.2 we review superbosonization. We complete this section with an overview of the generalized Hubbard-Stratonovich transformation in section 2.4.3.

2.4.1 Supermathematics

For a comprehensive approach to supersymmetry based on rigorous mathematics we refer to Ref. [115–119]. For approaches based on calculation rather than rigorous mathematics we refer to the physical literature [65,108,114,120–122]. We orientate our summary to the physical literature, because the formalism developed there is more intuitive when doing calculations.

Grassmann Variables

Supermathematics is concerned with the analysis and the algebra of domains consisting of commuting and anti-commuting variables. Commuting variables are numbers $x_i, x_j \in \mathbb{R}, \mathbb{C}$ with the property

$$x_i x_j = x_j x_i\ . \tag{2.45}$$

Contrarily, anti-commuting or Grassmann variables ζ_a, ζ_b satisfy

$$\zeta_a \zeta_b = -\zeta_b \zeta_a\ . \tag{2.46}$$

Importantly, Eq. (2.46) implies that the square of a anti-commuting variable vanishes $\zeta_a^2 = 0$. The variables ζ_a do not have a representation as an ordinary number. An associative algebra $\mathfrak{U}_q$ consisting of the q generators $\zeta_1, \cdots, \zeta_q$ satisfying Eq. (2.46) and a unit element is called a Grassmann algebra. Because of Eq. (2.46), all elements in the algebra $\mathfrak{U}_q$ are linear combinations of the unit element and $\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_l}$ for $1 \leq i_1 < i_2 < \cdots < i_l \leq q$. A prominent example of such an algebra is the exterior algebra generated by the basis elements of the cotangent space (the momentum space) in classical mechanics.

We introduce the complex conjugate ζ^* of a Grassmann variable ζ as an independent Grassmann variable such that

$$(\zeta\zeta^*)^* = \zeta\zeta^* \tag{2.47}$$

is "real", *i.e.* invariant under complex conjugation. In the case of commuting variables $(zz^*)^* = zz^*$ is enough to fix complex conjugation. For anti-commuting variables, it does not uniquely determine complex conjugation. We can either choose $\zeta^{**} = \zeta$ and $(\zeta_1\zeta_2)^* = \zeta_1^*\zeta_2^*$ or

$$\zeta^{**} = -\zeta \quad \text{and} \quad (\zeta_1\zeta_2)^* = \zeta_2^*\zeta_1^* \ . \tag{2.48}$$

We choose Eq. (2.48) in this thesis.

On complex or real Grassmann algebras $\mathfrak{U}_q$ we can have functions $f : \mathfrak{U}_q \to \mathfrak{U}_q$. Because of $\zeta_i^2 = 0$, every function is a polynomial

$$f(\zeta_1, \ldots, \zeta_q, \zeta_1^*, \ldots, \zeta_q^*) = \sum_{I \in \{0,1\}^{2q}} f_I \zeta_1^{i_1} \cdots \zeta_q^{i_q} \zeta_1^{*i_{q+1}} \cdots \zeta_q^{*i_{2q}} \ , \tag{2.49}$$

where $I = (i_1, \ldots, i_{2q})$ and $f_I \in \mathbb{R}, \mathbb{C}$. The exponential

$$\exp(a\zeta\zeta^*) = 1 + a\zeta\zeta^* = \frac{1}{1 - a\zeta\zeta^*} \ , \tag{2.50}$$

is a common example occurring in many sections later on.

Supervectors and Supermatrices

We introduce a $(p|q)$ dimensional supervector as a vector with p commuting entries z_i and q anti-commuting entires ζ_a,

$$\Psi = \begin{bmatrix} z_i \\ \zeta_a \end{bmatrix} \ . \tag{2.51}$$

The notation $(p|q)$ indicates, that the vector has p commuting and q anti commuting entires, also referred to as bosonic and fermionic dimensions. Both the fermionc as well as the bosonic entries of the supervector can be real or complex in which case we call Ψ a real or complex supervector, respectively. The space $\mathbb{K}^{p|q} = \mathbb{K}^p \times \mathfrak{U}_q$ consisting of all $(p|q)$ dimensional supervectors is called the $(p|q)$ dimensional supervector space or superdomain. The transposition of a vector is the same as in the ordinary case

$$\Psi^T = [\ z_i \mid \zeta_a\ ] \ . \tag{2.52}$$

For two complex supervectors we introduce the scalar product as

$$\Psi_1^\dagger \Psi_2 = \sum_{i=1}^{p} z_{i1}^* z_{i2} + \sum_{a=1}^{q} \zeta_{a1}^* \zeta_{a2} \ . \tag{2.53}$$

Analogous to the action of matrices in an ordinary vector space, the action of supermatrices in supervector spaces is introduced. In our particular choice of basis a matrix acting on a $(p|q)$-dimensional supervector space is of the form

$$\sigma = \begin{bmatrix} a & \nu \\ \mu & b \end{bmatrix} \ . \tag{2.54}$$

Here a and b are $p\times p$, respectively, $q\times q$ matrices with commuting entries. Whereas ν and μ are $p\times q$ and $q\times p$ dimensional matrices with anti-commuting entries. The individual blocks a, b, ν and μ are the so called boson-boson, fermion-fermion, boson-fermion and fermion-boson blocks. These names have their origin in physics. For instance, the boson-fermion block maps roughly speaking a fermionic particle, which is anti-commuting, to the space of commuting particles, the bosons.

The supermatrix σ, as introduced in Eq. (2.54), maps $\mathbb{K}^{p|q}$ on itself. Besides these, we have linear mappings from $\mathbb{K}^{p|q}$ to $\mathbb{K}^{m|n}$, represented as supermatrices. Then the corresponding supermatrix is of the form (2.54), with a, b being $m\times p$, $n\times q$ matrices with commuting entries, whereas μ and ν are $m\times q$ and $n\times p$ matrices with anti-commuting entries. The dimension of a generic supermatrix is stated as $(m|n)\times(p|q)$. This notation indicates that a $(p|q)$-dimensional supervector is mapped to a $(m|n)$-dimensional one. We call a supermatrix with $m=p$ and $n=q$ a square supermatrix. Otherwise it is a rectangular supermatrix. By $\mathbf{1}_{p|q}$ we denote the $(p|q)\times(p|q)$-dimensional unit matrix.

Multiplication of supermatrices is the same as for ordinary matrices. Very important transformations of a supermatrix are the transposition and the Hermitian conjugation. Their definitions differ from transposition and Hermitian conjugation of an ordinary matrix. They are introduced as

$$\sigma^T = \begin{bmatrix} a^T & -\mu^T \\ \nu^T & b^T \end{bmatrix} \quad \text{and} \quad \sigma^\dagger = (\sigma^*)^T \ , \tag{2.55}$$

such that $(\sigma_1\sigma_2)^T = \sigma_2^T\sigma_1^T$ and analogously for the Hermitian conjugation. Importantly, this definition makes sure that Hermitian conjugation is an involution so that $\sigma^{\dagger\dagger} = \sigma$.

A class of matrices that will become important later are the Hermitian (self-adjoint) supermatrices, *i.e.* square supermatrices σ as introduced in Eq. (2.54), invariant under Hermitian conjugation

$$\sigma = \begin{bmatrix} a & \nu \\ -\nu^\dagger & b \end{bmatrix} \ , \tag{2.56}$$

where $a = a^\dagger,\ b = b^\dagger$. Self-adjoint supermatrices occur when the supersymmetry method is applied to Hermitian random matrix ensembles like the complex ($\beta = 2$) Wishart model. Supermatrices occurring for the remaining symmetry classes have additional symmetries, as shown in Eq. (2.84).

Most concepts of ordinary vectors and matrices were extended to the theory of supervectors and supermatrices [115]. Important examples are the supertrace and the superdeterminant of a supermatrix. For a supermatrix σ as introduce in Eq. (2.54), the supertrace is given by

$$\mathrm{str}\sigma = \mathrm{tr}a - \mathrm{tr}b \ . \tag{2.57}$$

Analogous to the ordinary trace the supertrace is invariant under permutations $\mathrm{str}\sigma_1\sigma_2 = \mathrm{str}\sigma_2\sigma_1$. Another important function on the space of ordinary square matrices is the determinant. This concept was extended to the superdeterminant

$$\text{sdet}\sigma = \frac{\det(a-\nu b^{-1}\mu)}{\det b} = \frac{\det a}{\det(b-\mu a^{-1}\nu)}, \tag{2.58}$$

whenever a, b are invertible. In the way it is introduced in Eq. (2.58) it has the property that $\text{sdet}\sigma_1\sigma_2 = \text{sdet}\sigma_1\,\text{sdet}\sigma_2$. The connection between the superdeterminant and the supertrace is the same as the connection of the determinant and the trace,

$$\text{sdet}\exp(\sigma) = \exp(\text{str}\sigma) \ , \tag{2.59}$$

where the exponential of a supermatrix is defined by its series expansion.

Supergroups

In linear algebra the classical groups O(N), U(N) and USp$(2N)$ play a central role. They can be introduced by considering a complex N dimensional vector space V equipped with a Hermitian bilinear form $h(v,w) = v^\dagger w$. This means, $h(v,w) = h^*(w,v)$. Those matrices leaving h invariant and are invertible constitute U(N). Since h is a Hermitian bilinear form, Reh is a symmetric and Imh a skew-symmetric bilinear form. The group of matrices leaving invariant Reh and Imh is the orthogonal group O$(2N)$ and the unitary-symplectic group USp$(2N)$, respectively. Moreover, the subgroup of matrices $U \in$ U(N) with the property $U^* = U$ is equivalent to O(N).

We extend this definition to supervector spaces and introduce supergroups. We will only summarize the salient features of these groups, important for later purpose. To begin with, we take Ψ to be a complex supervector of dimension $(p|q)$ as introduced in Eq. (2.51). The linear transformations leaving invariant the Hermitian bilinear form $\Psi^\dagger\Psi$ constitute the group of unitary supermatrices,

$$\text{U}(p|q) = \left\{ u \middle| u^\dagger u = \mathbf{1}_{p|q} \right\} \ . \tag{2.60}$$

The largest ordinary subgroup of U$(p|q)$ is U$(p)\times$U(q). Similar to the case of the ordinary vector space, we the obtain two other non-trivial supergroups by considering the bilinear form $\Psi^\dagger\Psi$. We separate it into its real and imaginary part

$$\Psi^\dagger\Psi = \frac{1}{2}\left(\Psi^\dagger\Psi + \Psi^T\Psi^*\right) + \frac{1}{2}\left(\Psi^\dagger\Psi - \Psi^T\Psi^*\right) = \text{Re}\Psi^\dagger\Psi + \imath\text{Im}\Psi^\dagger\Psi \ . \tag{2.61}$$

The real part of $\Psi^\dagger\Psi$ is symmetric in the commuting entries (the z's) and skew-symmetric in the anti-commuting entires (the $\zeta's$). For the imaginary part it is reversed. The group leaving invariant the real and the imaginary part of $\Psi^\dagger\Psi$ is the unitary-ortho-symplectic group UOSp$^{(+)}(2p|2q)$ and UOSp$^{(-)}(2p|2q)$ with the largest ordinary subgroup given by O$(2p)$ $\times$ USp$(2q)$ and USp$(2p)$ $\times$ O$(2q)$, respectively. For the shake of completeness we mention that, similar to O$(N) \subset$ U(N), we obtain the two unitary-ortho-symplectic subgroups UOSp$^{(+)}(p|2q)$ and UOSp$^{(-)}(2p|q)$ of U$(2p|2q)$.

As in the ordinary case, not only compact but also non-compact supergroups do exist. In the unitary case we introduce it as the group of supermatrices leaving invariant the pseudo scalar product $\Psi^\dagger L\Psi$, where L is a diagonal matrix with ± 1 entries only. If $L = \mathrm{diag}(\mathbf{1}_{p-n}, -\mathbf{1}_n, \mathbf{1}_{q-m}, -\mathbf{1}_m)$ we denote the group of pseudo-unitary supermatrices by $\mathrm{U}(p-n, n|q-m, m)$. If the particular signature of L is unknown, we write $\mathrm{U}(L)$. Analogously to the non-compact case we introduce $\mathrm{UOSp}^{(+)}(L)$ and $\mathrm{UOSp}^{(-)}(L)$ as the supergroups leaving invariant the real and the imaginary part of $\Psi^\dagger L\Psi$, respectively.

In the theory of ordinary matrices, we can diagonalize Hermitian matrices such that $H = UXU^\dagger$, where $U \in \mathrm{U}(N)$ and $X = \mathrm{diag}(x_1, \ldots, x_N)$ are the real eigenvalues. In a similar fashion we can diagonalize a Hermitian supermatrix $\sigma = usu^\dagger$, where $u \in \mathrm{U}(p|q)$ and $s = \mathrm{diag}(s_{11}, \ldots, s_{1p}, s_{12}, \ldots, s_{q2})$ are the eigenvalues. We separate s into the bosonic eigenvalues s_{i1} and fermionic eigenvalues s_{i2}.

The $(2p|2q) \times (2p|2q)$ dimensional supermatrix σ obtained by superbosonization and the generalized Hubbard-Stratonovich transformation for either $\beta = 1$ or $\beta = 4$ is diagonalized by $u \in \mathrm{UOSp}^{(+)}(2p|2q)$ or $u \in \mathrm{UOSp}^{(-)}(2p|2q)$, respectively.

Superanalysis

From the algebra we switch to the analysis on superdomains. Because of Eq. (2.46), any function $f(z,\zeta)$ on a superdomain $\mathbb{K}^{p|q}$ is a polynomial in the Grassmann variables with smooth coefficients in z,

$$f(z,\zeta) = \sum_{I\in\{0,1\}^{2q}} f_I(z)\zeta_1^{i_1}\cdots\zeta_q^{i_q}\zeta_1^{*i_{q+1}}\cdots\zeta_q^{*i_{2q}} \ , \tag{2.62}$$

where z and ζ are vectors and $I = (i_1, \ldots, i_{2q})$. As in ordinary analysis we introduce the derivative of a function. The derivative of $f(z,\zeta)$ with respect to the commuting variables z_i is the ordinary derivative acting on all the f_I. For the anti-commuting variables, the definition has a subtlety. Due to the anti-commuting property, we can define either a derivative acting from the right or from the left. The resulting expressions differ by a sign. We stick to the left derivative. Since we have polynomials only, it is enough do define it by its action on a monomials in $\mathfrak{U}_q$

$$\frac{\partial}{\partial\zeta_i}\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_l} = \delta_{ii_1}\zeta_{i_2}\cdots\zeta_{i_l} - \delta_{ii_2}\zeta_{i_1}\zeta_{i_3}\cdots\zeta_{i_l} + \ldots \ , \tag{2.63}$$

The derivative is itself anti-commuting. Similarly to ordinary calculus, we have a product as well as a chain rule [115].

We employ the derivative with respect to the Grassmann variables and Taylor expand $f(z,\zeta)$ around $\zeta_a = \zeta_a^* = 0$ for all a. We obtain that

$$f_I(z) = \left.\frac{\partial^{|I|}}{\partial\zeta_q^{*i_{2q}}\cdots\partial\zeta_1^{*i_{q+1}}\partial\zeta_q^{i_q}\cdots\partial\zeta_1^{i_1}} f(z,\zeta)\right|_{\zeta_i=\zeta_i^*=0} . \tag{2.64}$$

where $f_I(z)$ is as introduced in Eq. (2.62) and $|I| = i_1 + \cdots + i_{2q}$.

Another vital operation in analysis is the integration. Since anti-commuting variables can not be represented as a number, the integral over a Grassmannian does not correspond to a volume. The definition of the integral is a formal one. It is known as Berezin integral [115,123] and is defined by

$$\int \mathrm{d}\zeta_a \zeta_b = \frac{\delta_{ab}}{\sqrt{2\pi}} \quad \text{and} \quad \int \mathrm{d}\zeta_a = 0 \ . \tag{2.65}$$

The choice of normalization $1/\sqrt{2\pi}$ is common and will become clear later. Analogously to the derivatives with respect to Grassmannian variables, the differentials $\mathrm{d}\zeta_a$ are anti-commuting. Some authors define the integral (2.65) using the derivative with respect to Grassmannians,

$$\int \mathrm{d}[\zeta] \; f(\zeta) = \frac{1}{\sqrt{2\pi}^q} \prod_{a=1}^{q} \frac{\partial}{\partial \zeta_a} f(\zeta) \ , \tag{2.66}$$

where $\mathrm{d}[\zeta] \ = \prod_{a=1}^{q} \mathrm{d}\zeta_a$. We continue to use the integral notation such that we can avoid using left and right derivatives. Furthermore, it is more convenient when doing coordinate changes.

From the definition (2.65) it follows that the integral over all elements of the Grassmann algebra $\mathfrak{U}_q$ is proportional to the element $f_{11\ldots1}(z)$ in the expansion (2.62). Similar to ordinary integration, we have partial integration and we can do coordinate changes. Suppose we integrate a function $f(a\zeta)$ and do the coordinate change $\zeta' = a\zeta$. Under the integral this coordinate change yields $\mathrm{d}\zeta' = \mathrm{d}\zeta/a$. Obviously, this is different from ordinary integration, where a substitution of the form $y' = ay$ would lead to $\mathrm{d}y' = a\mathrm{d}y$. When integrating a function over a vector with Grassmannian entries, a coordinate change of the form $\eta = A\zeta$, where A is an ordinary, invertible matrix and η, ζ are vectors with Grassmannian entries, yields $\mathrm{d}[\zeta] \ = \det^{-1} A \ \mathrm{d}[\zeta]$.

Two prominent examples of integrals over Grassmannian vectors are the real and complex Gaussian integrals. We assume that A is a anti-symmertic, even dimensional matrix and obtain that [104]

$$\int \mathrm{d}[\zeta] \ \exp\left(\zeta^T A \zeta\right) = \mathrm{pf}\frac{A}{\sqrt{2\pi}} \ , \tag{2.67}$$

where where pf is the Pfaffian determinant and ζ is a vector with real Grassmannian entries. If ζ is a vector with complex Grassmannian entries and A an $q \times q$-dimensional Hermitian matrix, we arrive at

$$\int \mathrm{d}[\zeta] \ \exp\left(\zeta^\dagger A \zeta\right) = \det\frac{A}{2\pi} \ , \tag{2.68}$$

where $\mathrm{d}[\zeta] \ = \prod_{a=1}^{q} \mathrm{d}\zeta_a \mathrm{d}\zeta_a^*$. The $1/2\pi$ is due to our choice of normalization in Eq. (2.65) and therefore justified it retrospectively. Hence, in contrast to the ordinary complex Gaussian integral which leads to an determinant of the inverse of A, we obtain a determinant of A.

We combine the integration over $\mathbb{K}^p$ with the integration over $\mathfrak{U}_q$ to an integral over the superdomain $\mathbb{K}^{p|q}$,

$$\int \mathrm{d}[\Psi]\ f(\Psi)\ , \tag{2.69}$$

where $\Psi = (z_1, \ldots, z_p, \zeta_1, \ldots, \zeta_q)$ and $\mathrm{d}[\Psi]\ = \mathrm{d}[z]\ \mathrm{d}[\zeta]$. A prime example of an integral over a superdomain is the supersymmetric extension of the Gaussian integral

$$\int \mathrm{d}[\Psi]\ \exp\left(\imath\Psi^\dagger\sigma\Psi\right) =\ \mathrm{sdet}^{-1}\sigma, \tag{2.70}$$

where σ is an invertible Hermitian supermatrix and Ψ is a complex supervector. A simultaneous coordinate change in the fermionic and bosonic sector, *i.e.* the anticommuting and commuting part of the superdomain, leads to a coordinate change in the whole superdomain, *i.e.* $\Psi \mapsto \Psi(\chi)$. We take the variables z, ζ to be the coordinates describing Ψ and w, η the coordinates describing χ. Under the integral this coordinate change leads to the following change of the integration measure

$$\mathrm{d}[\chi]\ =\ \mathrm{sdet}\frac{\partial\chi}{\partial\Psi^T}\mathrm{d}[\Psi]\ =\ \mathrm{sdet}\begin{bmatrix} \dfrac{\partial w}{\partial z^T} & \dfrac{\partial w}{\partial \zeta^T} \\ \dfrac{\partial \eta}{\partial z^T} & \dfrac{\partial \eta}{\partial \zeta^T} \end{bmatrix}\mathrm{d}[\Psi]\ \ . \tag{2.71}$$

The Jacobian caused by the coordinate transformation is in supermathematics known as Berezinian. The transformation of the measure as introduced in Eq. (2.71) has to be treated with caution, because it is valid for integrals over compactly supported functions only [115]. For integrals over non-compactly supported functions the mixing of ordinary and Grassmann variables can lead to so called Efetov-Wegner or Rothstein terms [65, 124].

To conceive these "correction" terms, we follow Ref. [115]. We take $v(x) > 0$ to be the defining inequality of the support of $f(x, \zeta)$. Hence, the boundary of the support is given by $v(x) = 0$. We do the coordinate change $\Psi = (x, \zeta) \mapsto \Psi(\chi) = (x(y, \eta), \zeta(y, \eta))$. With the aid of a Θ-function, we perform a coordinate change as described above yielding [115]

$$\int \mathrm{d}[\Psi]\ f(x, \zeta) = \int \mathrm{d}[\chi]\ \ \mathrm{sdet}\frac{\partial\Psi}{\partial\chi^T}\ f(x(y, \eta), \zeta(y, \eta))\Theta(v(x(y, \eta)))\ . \tag{2.72}$$

The correction terms to Eq. (2.71) are achieved by expanding the Θ-function in terms of Grassmann variables. Since these additional terms consist of derivatives of the Heaviside function, all correction terms involve δ-functions and derivatives thereof.

2.4.2 Superbosonization

We briefly review the method of superbosonization and shed light on its application in the context of random matrix theory. For the mathematical details and the proofs we refer to Ref. [125]. To begin with, we take z and ζ to be $N \times p$ and

$N \times q$ dimensional matrices with commuting and anti-commuting complex entires, respectively. We aim at integrating functions $f(z, z^\dagger, \zeta, \zeta^\dagger)$ depending on these variables,

$$\int \mathrm{d}[z]\ \mathrm{d}[\zeta]\ f(z, z^*, \zeta, \zeta^*)\ . \tag{2.73}$$

The function $f(z, z^*, \zeta, \zeta^*)$ is supposed to be holomorphic in z, z^*, ζ and ζ^* and invariant under the action of $g \in \mathrm{G}_N$, *i.e.* $f(gz, z^* g^{-1}, g\zeta, \zeta^* g^{-1}) = f(z, z^*, \zeta, \zeta^*)$.

We begin with $\mathrm{G}_N = \mathrm{U}(N)$. Under these moderate assumptions on f, the authors proofed the existence of a function $F(x, y, \eta, \tau)$ holomorpic in the commuting variables $x_{cc'}, y_{ee'}$ and the anti-commuting variables $\eta_{ce'}, \tau_{ec'}$, where $c, c' = 1, \ldots, p$ and $e, e' = 1, \ldots, q$, which we write as

$$F(x, y, \eta, \tau) = F \begin{pmatrix} x & \eta \\ \tau & y \end{pmatrix} = F(\sigma) \tag{2.74}$$

such that

$$F\left(A^\dagger A\right) = f(z, z^*, \zeta, \zeta^*)\ . \tag{2.75}$$

Here we introduce the rectangular $(p|q) \times (N|0)$-dimensional supermatrix

$$A = [\ z_{ia} \mid \zeta_{ib}\ ]\,. \tag{2.76}$$

With the aid of the function F, it was shown that the integral (2.73) can be written as an integral over a $(p|q) \times (p|q)$-dimensional supermatrix σ,

$$\int \mathrm{d}[z]\ \mathrm{d}[\zeta]\ f(z, z^*, \zeta, \zeta^*) \sim \int_D \mathrm{D}\sigma F(\sigma) \mathrm{sdet}^N \sigma\ F(\sigma)\ . \tag{2.77}$$

The domain of integration is $D = D_p^0 \times D_q^1$, with D_p^0 the space of positive definite Hermitian matrices of dimension $p \times p$ and D_q^1 the group of $q \times q$ unitary matrices. The measure $\mathrm{D}\sigma$ used in Eq. (2.77) is the Berezin superintegral form given by [115]

$$\mathrm{D}\sigma \sim \mathrm{d}\mu_{D_p^0}(y) \mathrm{d}\mu_{D_q^1}(x) \prod_{\substack{1 \le c \le q \\ 1 \le e \le q}} \frac{\partial^2}{\partial \eta_{ce} \partial \tau_{ec}}\ \mathrm{det}^q(x - \eta y^{-1} \tau) \mathrm{det}^p(y - \tau x^{-1} \eta)\ , \tag{2.78}$$

where the invariant measures in the boson-boson and the fermion-fermion block are

$$\mathrm{d}\mu_{D_q^1}(x) \sim \mathrm{det}^{-q} x \mathrm{d}[x] \quad \text{and} \quad \mathrm{d}\mu_{D_p^0}(y) \sim \mathrm{det}^{-p} y \mathrm{d}[y]\ , \tag{2.79}$$

respectively. The measure $\mathrm{d}\mu_{D_q^1}(x)$ is the Haar-measure on the space of unitary $p \times p$ matrices.

If the function $f(z, z^*, \zeta, \zeta^*)$ is invariant under $\mathrm{G}_N = \mathrm{O}(N)$, $\mathrm{USp}(2N)$ the analysis of Ref. [125] shows that we can find a non-unique function F such that

$$F\left(A^\dagger A\right) = f(z, z^*, \zeta, \zeta^*)\ , \tag{2.80}$$

where $A^\dagger A$ is a $(2q|2p) \times (2q|2p)$-dimensional Hermitian supermatrix. For $\mathrm{G}_N = \mathrm{O}(N), \mathrm{USp}(2N)$ the rectangular $(2q|2p) \times (\gamma_2 N|0)$-dimensional supermatrix A has a different structure compared to those considered previously. It is due to additional symmetries in the underlying vector space such that for $\mathrm{O}(N)$ and $\mathrm{USp}(2N)$ the supermatrix A is given by

$$A = \left[\ z_{ja} \mid z_{ja}^* \mid \zeta_{jb} \mid \zeta_{jb}^*\ \right], \tag{2.81}$$

and

$$A = \left[\begin{array}{c|c|c|c} z_{2j,a} & -z_{2j+1,a}^* & \zeta_{2j,b} & -\zeta_{2j+1,b}^* \\ \hline z_{2j+1,a} & z_{2j,a}^* & \zeta_{2j+1,b} & \zeta_{2j,b}^* \end{array}\right], \tag{2.82}$$

respectively. The structure of A carries over to $A^\dagger A$ in terms of an additional symmetry. If we introduce the matrices

$$T_{\mathrm{O}(N)} = \begin{bmatrix} 0 & \mathbf{1}_p & 0 & 0 \\ \mathbf{1}_p & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1}_q \\ 0 & 0 & \mathbf{1}_q & 0 \end{bmatrix} \quad \text{and} \quad T_{\mathrm{USp}(2N)} = \begin{bmatrix} 0 & -\mathbf{1}_p & 0 & 0 \\ \mathbf{1}_p & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_q \\ 0 & 0 & \mathbf{1}_q & 0 \end{bmatrix}, \tag{2.83}$$

we obtain that $A^\dagger A$ has the following symmetry

$$A^\dagger A = T_{\mathrm{O}(N)} (A^\dagger A)^T T_{\mathrm{O}(N)}^{-1} \quad \text{and} \quad A^\dagger A = T_{\mathrm{USp}(2N)} (A^\dagger A)^T T_{\mathrm{USp}(2N)}^{-1} . \tag{2.84}$$

A supermatrix σ replacing the z, ζ integrals in Eq. (2.73) has to possess the same symmetry as $A^\dagger A$ and therefore has to satisfy Eq. (2.84). It has two commuting and two anti-commuting blocks x, y and η, τ. The symmetry (2.84) restricts x and y to a particular subset of Hermitian and unitary matrices, whereas in the anti-commuting blocks it identifies the entries of τ with those of η. As shown by the authors in Ref. [125], the integral (2.73) with f being invariant under the action of $\mathrm{G}_N = \mathrm{O}(N)$ or $\mathrm{G}_N = \mathrm{USp}(2N)$, can be written as an integral over an $(2q|2p) \times (2q|2p)$-dimensional supermatrix,

$$\int \mathrm{d}[z]\ \mathrm{d}[\zeta]\ f(z, z^*, \zeta, \zeta^*) \sim \int_D \mathrm{D}\sigma \,\mathrm{sdet}^{N/2} \sigma F(\sigma) . \tag{2.85}$$

We discuss the domain of integration D case by case. For $\beta = 1$ the domain of integration is invariant under the symmetry transformation on the left hand side of Eq. (2.84). Thus, in an appropriate basis, the boson-boson and the fermion-fermion block consists of a real symmetric, positive definite matrix and a matrix in the circular symplectic ensemble $\mathrm{U}(2q)/\mathrm{USp}(2q)$, respectively. For $\beta = 4$ the domain of integration is invariant under the left side of Eq. (2.84). Hence, the boson-boson and the fermion-fermion block consists of a real quaternion, self-dual, positive definite matrix and a matrix in the circular orthogonal ensemble $\mathrm{U}(2q)/\mathrm{O}(2q)$. Analogously to the $\mathrm{U}(N)$ case, the measure $\mathrm{D}\sigma$ in Eq. (2.85) is the Berezin superintegral form on the particular space. For $\beta = 1$ it is given by

$$\begin{aligned}\mathrm{D}\sigma \sim\, & \mathrm{d}\mu_{D_p^0}(y)\mathrm{d}\mu_{D_q^1}(x) \prod_{c=1,e=1}^{p,q} \frac{\partial^2}{\partial\eta_{ce}\partial\eta_{(e+q)(c+p)}} \prod_{c=1,e=1}^{p,q} \frac{\partial^2}{\partial\eta_{e(c+p)}\partial\eta_{c(e+q)}} \\ & \times \frac{\det^q(x-\sigma y^{-1}\tau)\det^p(y-\tau x^{-1}\sigma)}{\det^{1/2}(\mathbf{1}_p - x^{-1}\sigma y^{-1}\tau)} \ .\end{aligned} \tag{2.86}$$

For $\beta = 4$ the derivatives $\partial\eta_{e(c+p)}$ and $\partial\eta_{c(e+q)}$ in the second product are interchanged and the exponent of the determinant in the denominator becomes $-1/2$. The invariant measures in the boson-boson and the fermion-fermion sectors are

$$\mathrm{d}\mu_{D_q^1}(x) \sim \det{}^{-q-1/2}x\mathrm{d}[x] \quad , \quad \mathrm{d}\mu_{D_p^0}(y) \sim \det{}^{-p+1/2}y\mathrm{d}[y] \quad , \tag{2.87}$$

and

$$\mathrm{d}\mu_{D_q^1}(x) \sim \det{}^{-q+1/2}x\mathrm{d}[x] \quad , \quad \mathrm{d}\mu_{D_p^0}(y) \sim \det{}^{-p-1/2}y\mathrm{d}[y] \quad , \tag{2.88}$$

for $\beta = 1, 4$ respectively.

2.4.3 Generalized Hubbard-Stratonovich Transformation

The generalized Hubbard-Stratonovich transformation [109, 113] is an alternative method to superbosonization. It has been shown that both approaches are equivalent [126]. Although it is not based on rigorous mathematics, it has the advantage that a replacement of a supermatrix using a δ-function is more intuitive to physicists and applicable to a broader class of matrix models, *c.f.* Ref. [8].

As in the previous section, the Hubbard-Stratonovich transformation is used to replace a dyadic product of two rectangular supermatrices by a square supermatrix. We take z and ζ to be a $N \times p$ and a $N \times q$ matrix with complex commuting and anti-commuting entries, respectively and aim to investigate

$$\int \mathrm{d}[z]\ \mathrm{d}[\zeta]\ f(z, z^*, \zeta, \zeta^*) \ . \tag{2.89}$$

The integrand f is holomorphic in z, z^*, ζ and ζ^* as well as invariant under the action of G_N. We begin with the case $\mathrm{G}_N = \mathrm{U}(N)$. As shown in Ref. [125], a function $F(x, y, \eta, \tau)$ exists that is holomorpic in the commuting variables $x_{cc'}, y_{ee'}$ and the anti-commuting variables $\eta_{ce'}, \tau_{ec'}$, which we write as

$$F(x, y, \eta, \tau) = F\begin{pmatrix} x & \eta \\ \tau & y \end{pmatrix} = F(\sigma) \ , \tag{2.90}$$

where e, c', c and c' are as defined above, such that

$$F\left(A^\dagger A\right) = f(z, z^*, \zeta, \zeta^*) \ . \tag{2.91}$$

Here A is a rectangular $(p|q) \times (N|0)$-dimensional supermatrix as introduced in Eq. (2.76) so that $A^\dagger A$ is a $(p|q) \times (p|q)$ dimensional supermatrix. Because of the dyadic structure $A^\dagger A$ is a Hermitian supermatrix. We take the Fourier transform of a δ-function in superspace [127]

$$\delta(\sigma - \mu) \sim \int \mathrm{d}[\rho] \ \exp\left(\imath \mathrm{str}\rho(\sigma - \mu)\right) \ , \tag{2.92}$$

and replace $A^\dagger A$ by a Hermitian supermatrix σ yielding

$$\begin{aligned} &\int \mathrm{d}[z] \ \mathrm{d}[\zeta] \ f(z, z^*, \zeta, \zeta^*) \\ =&\int \mathrm{d}[z] \ \mathrm{d}[\zeta] \ \mathrm{d}[\sigma, \rho] \ F(\sigma) \exp\left(\imath \mathrm{str}\rho^-(\sigma - A^\dagger A)\right) \ . \end{aligned} \tag{2.93}$$

We introduce $\rho^- = \rho - \imath\epsilon \mathbf{1}_{p|q}$ and keep in mind that $\epsilon \to 0$ subsequently to the calculation of the ρ integral. This imaginary increment ensures the convergence, when exchanging the z and ζ with the σ and ρ integrals. Moreover, we assume a proper rotation of σ

$$\sigma = \mathrm{diag}(\mathbf{1}_p, e^{\imath\phi/2}\mathbf{1}_q)\sigma_0 \mathrm{diag}(\mathbf{1}_p, e^{-\imath\phi/2}\mathbf{1}_q) \ , \tag{2.94}$$

where σ_0 is of the form Eq. (2.56), such that the resulting σ integral exists. Due to the exchange of the integrals, the A integral reduces to a Gaussian integral. To perform it, we use Eq. (2.70) and find

$$\int \mathrm{d}[z] \ \mathrm{d}[\zeta] \ \exp\left(-\imath \mathrm{str}\rho^- A^\dagger A\right) \sim \mathrm{sdet}^{-N} \rho^- \ . \tag{2.95}$$

Substituting the superdeterminant (2.95) into the integral (2.93), we arrive at

$$\int \mathrm{d}[z] \ \mathrm{d}[\zeta] \ f(z, z^*, \zeta, \zeta^*) = \int \mathrm{d}[\sigma] \ F(\sigma) I_N(\sigma) \ . \tag{2.96}$$

Here I_N is the supersymmetric Ingham-Siegel integral, which is given for $\beta = 1, 2, 4$ by

$$I_N(\sigma) = \int \mathrm{d}[\rho] \ \mathrm{sdet}^{-N/\gamma_1} \rho^- \exp\left(\imath \mathrm{str}\rho^- \sigma\right) \ , \tag{2.97}$$

where ρ is in the same symmetry class as σ, and γ_1 as introduced in section 2.2. Using the invariance of the flat measure $\mathrm{d}[\rho]$ under the action $\rho \mapsto u\rho u^\dagger$ with $u \in \mathrm{UOSp}^{(+)}(2p|2q)$, $u \in \mathrm{U}(p|q)$, $u \in \mathrm{UOSp}^{(-)}(2p|2q)$ for $\beta = 1, 2, 4$, respectively, it turns out that $I_N(\sigma)$ is invariant under the same action on σ. Thus, it depends on the eigenvalues of σ only. In terms of these $I_N(\sigma)$ was computed for $\beta = 1, 2, 4$ in Ref. [113].

For the remaining two symmetry class with $\mathrm{G}_N = \mathrm{O}(N), \mathrm{USp}(2N)$ the generalized Hubbard-Stratonovich transformation works in an analogous manner. The main difference is that $A^\dagger A$ is not only Hermitian but also invariant under the transformation (2.84). Thus, we replace the $(2p|2q) \times (2p|2q)$-dimensional $A^\dagger A$ by a supermatrix with the same symmetries using a δ-function, exchange the integrals and perform the A integral. This leads to

$$\int \mathrm{d}[z]\ \mathrm{d}[\zeta]\ f(z, z^*, \zeta, \zeta^*) = \int \mathrm{d}[\sigma]\ F(\sigma) I_N(\sigma)\ , \tag{2.98}$$

where we assume a proper rotation of σ and ρ similar to Eq. (2.94). Because σ and $A^\dagger A$ are in the same symmetry class, the boson-boson and the fermion-fermion block are for $\mathrm{G}_N = \mathrm{O}(N)$ given by a real symmetric and a real quaternion self-dual matrix, respectively. For $\mathrm{G}_N = \mathrm{USp}(2N)$ it is the other way around.

2.5 The Supersymmetry Method in Correlated Ensembles

As a first result of this thesis, we work out the supersymmetry method for the correlated Wishart model, extending a discussion in the appendix of Ref. [2]. In section 2.3.1, we related the k-point correlation function, to its generating function, see Eq. (2.35). A generalization of this arises if we take a different number of determinants in the numerator and in the denominator and study

$$Z_p^{k_1/k_2}(\kappa) = \frac{1}{Z_p^{0/0}(0)} \int \mathrm{d}[W]\ P(W|\hat{\Lambda})\ \frac{\prod_{a=1}^{k_2} \det\left(WW^\dagger - \kappa_{a,2} \mathbf{1}_{\gamma_2 p}\right)}{\prod_{b=1}^{k_1} \det\left(WW^\dagger - \kappa_{b,1} \mathbf{1}_{\gamma_2 p}\right)}\ , \tag{2.99}$$

where $\kappa = \mathrm{diag}(\kappa_{11}, \kappa_{21}, \ldots, \kappa_{k_1 1}, \kappa_{12}, \ldots, \kappa_{k_2 2}) = \mathrm{diag}(\kappa_1, \kappa_2)$ and γ_1, γ_2 and $\tilde{\gamma}$ are as introduced in section 2.2.1. What the particular values the κ_{a1} and κ_{b2} are differs from case to case. For instance, if we consider the density we have $k_1 = k_2 = 1$, $\kappa_{11} = x + \imath\varepsilon + j$ and $\kappa_{12} = x + \imath\varepsilon$, see also Eq. (2.34). To ensure the convergence of the integral (2.99), we require that $\mathrm{Im}\ \kappa_{b,1} \neq 0$ for all b. For illustrating purpose, we assume that $\mathrm{Im}\ \kappa_{b,1} < 0$ for all b. The more general case works analogously, but is more technical. The normalization of this matrix model is chosen such that $Z_p^{k_1/k_2}(\kappa) \to 1$ for $\kappa \to 0$.

To begin with our analysis, we only assume that the Fourier transform of the distribution $P(W|\hat{\Lambda})$ exists and that $P(W|\hat{\Lambda})$ is invariant under right translation,

$$P(W|\hat{\Lambda}) \mapsto P(WU|\hat{\Lambda}) = P(W|\hat{\Lambda}) \tag{2.100}$$

for all $U \in \mathrm{G}_n$. To map Eq. (2.99) to superspace, we express the ratio of determinants as Gaussian integral over a superdomain using Eq. (2.70),

$$\begin{aligned}
\frac{\prod_{a}^{k_2} \det\left(WW^\dagger - \kappa_{a,2} \mathbf{1}_{\gamma_2 p}\right)}{\prod_{b}^{k_1} \det\left(WW^\dagger - \kappa_{b,1} \mathbf{1}_{\gamma_2 p}\right)} &= \mathrm{sdet}^{\gamma_2(n-p)} \kappa\ \mathrm{sdet}^{-1}\left(W^\dagger W \otimes \mathbf{1}_{2k} - \mathbf{1}_{\gamma_2 n} \otimes \kappa\right) \\
\sim \mathrm{sdet}^{\gamma_2(n-p)} \kappa \int \mathrm{d}[\Psi]\ \exp&\left(\imath \Psi^\dagger \left(W^\dagger W \otimes \mathbf{1}_{2k} - \mathbf{1}_{\gamma_2 n} \otimes \kappa\right) \Psi\right)
\end{aligned} \tag{2.101}$$

Notice that from the first to the second line of Eq. (2.101), we replaced $WW^\dagger$ by $W^\dagger W$. This is justified, because p of the n eigenvalues of $W^\dagger W$ coincide with those of $WW^\dagger$. The remaining $n-p$ are zero. Hence, the exchange yields the superdeterminant of κ to a power of $p-n$.

For an ordinary vector v we have the following identity $v^\dagger v = \mathrm{tr} vv^\dagger$. A similar identity exist for vectors ξ with Grassmann entries $\xi^\dagger\xi = -\mathrm{tr}\xi\xi^\dagger$, where the minus sign is caused by anti-commuting ξ and $\xi^\dagger$. We combine both identities and introduce the Hermitian matrix K with entries

$$K_{ij} = \sum_{a=1}^{k_1} z_{ia} z_{ja}^* - \sum_{b=1}^{k_2} \zeta_{ib}\zeta_{jb}^* \quad \Leftrightarrow \quad K = [\ z_{ja} \mid \zeta_{jb}\ ] \left[\frac{z_{ja}^*}{-\zeta_{jb}}\right] . \tag{2.102}$$

As introduced in Eq. (2.102), K is a $n\times n$ matrix with commuting entries. With K at hand we obtain that the exponential in the third line of Eq. (2.101) can be written as

$$\exp\left(\imath\Psi^\dagger\left(W^\dagger W\otimes \mathbf{1}_k - \mathbf{1}_{\gamma_2 n}\otimes\kappa\right)\Psi\right) = \exp\left(\imath \mathrm{tr} W^\dagger W K - \imath\Psi^\dagger \mathbf{1}_{\gamma_2 n}\otimes\kappa\Psi\right) . \tag{2.103}$$

If we exchange the W and the Ψ integral, the W integral reduces to a Fourier transform of the probability distribution $P(W|\hat\Lambda)$, the so called characteristic function

$$\Phi_n(K|\hat\Lambda) = \int \mathrm{d}[W]\ P(W|\hat\Lambda)\exp\left(\imath \mathrm{tr} W^\dagger W K\right) . \tag{2.104}$$

It is also known as moment generating functional [37, 41], because the derivatives with respect to K at $K=0$ yield moments of $W^\dagger W$ with respect to the probability distribution $P(W|\hat\Lambda)$. We substitute into the generating functions such that it becomes

$$Z_p^{k_1/k_2}(\kappa) = K_{n,p}\ \mathrm{sdet}^{\gamma_2(n-p)}\kappa \int \mathrm{d}[\Psi]\ \exp\left(-\imath\Psi^\dagger \mathbf{1}_{\gamma_2 n}\otimes\kappa\Psi\right)\Phi_n(K|\Lambda) , \tag{2.105}$$

where $K_{n,p}$ is the overall normalization constant. For $\beta=2$ the matrix $W^\dagger W$ does not provide further symmetries. For $\beta = 1,4$ the entries of W are real, respectively, real quaternion so that $W^\dagger W$ is real symmetric or self-dual. In this case the characteristic function Φ_n depends only on the part of K that respects this symmetry, see Eq. (2.104), and the complementary part of it drops out. Thus, it is reasonable to decompose K for $\beta=1$ into its symmetric and anti-symmetric and for $\beta=4$ into its self-dual and non self-dual part

$$K = \frac{1}{2}\left(K+K^T\right) + \frac{1}{2}\left(K-K^T\right) = \frac{1}{2}\left(K_+ + K_-\right) , \tag{2.106}$$

$$K = \frac{1}{2}\left(K+\Omega_n K^T\Omega_n^{-1}\right) + \frac{1}{2}\left(K-\Omega_n K^T\Omega_n^{-1}\right) = \frac{1}{2}\left(K_+ + K_-\right) , \tag{2.107}$$

where Ω_n is the symplectic unit

$$\Omega_n = \left[\begin{array}{cc} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{array}\right] . \tag{2.108}$$

If we set $K_+ = K$ for $\beta = 2$, the matrix K_+ respects the same symmetries as $W^\dagger W$. From this follows that the characteristic function (2.104) depends only on K_+.

By the assumption that $P(W|\hat{\Lambda})$ is invariant under the right action of G_n on W, which means $U \in \mathsf{G}_n$ acts by $W \mapsto WU$, it immediately follows that the characteristic function is invariant under the adjoint action of G_n on K_+, *i.e.* $K_+ \mapsto UK_+U^\dagger$ where $U \in \mathsf{G}_n$. Hence, it depends solely on the invariants of K_+, namely $\mathrm{tr}K_+^m$ for $m \in \mathbb{N}$.

In Eq. (2.76) we introduced the rectangular supermatrix A. As we have shown in Eq. (2.102), $K = AA^\dagger = K_+$. If we consider instead of K the matrix K_+, we obtain that

$$K_+ = AA^\dagger \tag{2.109}$$

holds for $\beta = 1, 2, 4$ with A given by Eqs. (2.81), (2.76), (2.82). Due to the dyadic structure of K_+, it has the same invariants as $A^\dagger A$. This is commonly known as trace duality [109, 113] and is written as

$$\mathrm{tr}K_+^m = \mathrm{tr}\left(AA^\dagger\right)^m = \mathrm{str}\left(A^\dagger A\right)^m \quad \text{for all } m \in \mathbb{N}\ . \tag{2.110}$$

If we substitute the identity (2.110) into the integrand (2.105), it depends on the supermatrix $A^\dagger A$ only, because

$$\Psi^\dagger \mathbf{1}_{\gamma_2 n} \otimes \kappa \Psi = 1/\tilde{\gamma}\ \mathrm{str}\hat{\kappa}A^\dagger A\ , \tag{2.111}$$

where $\hat{\kappa} = \mathbf{1}_{\tilde{\gamma}} \otimes \mathrm{diag}(\kappa_1, \kappa_2)$. Thus, we arrive at

$$Z_p^{k_1/k_2}(\kappa) = K_{n,p}\ \mathrm{sdet}^{\gamma_2(n-p)}\kappa \int \mathrm{d}[\Psi]\ \exp\left(-\frac{\imath}{\tilde{\gamma}}\mathrm{str}\hat{\kappa}A^\dagger A\right)\tilde{\Phi}_n(A^\dagger A/\tilde{\gamma}|\Lambda)\ , \tag{2.112}$$

where the function $\tilde{\Phi}_n$ is the characteristic function Φ_n with the invariant $\mathrm{tr}\left(AA^\dagger\right)^m$ replaced by $\mathrm{str}\left(A^\dagger A\right)^m$.

We change the coordinates and replace $A^\dagger A$ by a supermatrix in the same symmetry class. To do so, two equivalent approaches exist in the literature, the generalized Hubbard-Stratonovich transformation [109, 113] and superbosonization [125, 128], reviewed in section 2.4.3 and section 2.4.2. In the former, we replace $A^\dagger A$ by a Hermitian supermatrix σ in the same symmetry class and end up with

$$Z_p^{k_1/k_2}(\kappa) = K_{n,p}\ \mathrm{sdet}^{\gamma_2(n-p)}\kappa \int \mathrm{d}[\sigma]\ \exp\left(-\frac{\imath}{\tilde{\gamma}}\mathrm{str}\hat{\kappa}\sigma\right)\ \tilde{\Phi}_n(\sigma/\tilde{\gamma}|\Lambda)I_n(\sigma)\,, \tag{2.113}$$

where σ is a properly rotated supermatrix, see Eq. (2.94) and $I_n(\sigma)$ the supersymmetric Ingham-Siegel integral (2.97).

The second possibility to replace $A^\dagger A$ in Eq. (2.112) is superbosonization. Employing it as described in section 2.4.2 leads to

$$Z_p^{k_1/k_2}(\kappa) = K_{n,p}\ \mathrm{sdet}^{\gamma_2(n-p)}\kappa \int \mathrm{D}\sigma \mathrm{sdet}^{n/\tilde{\gamma}}\sigma \exp\left(-\frac{\imath}{\tilde{\gamma}}\mathrm{str}\hat{\kappa}\sigma\right)\ \tilde{\Phi}_n(\sigma/\tilde{\gamma}|\Lambda), \tag{2.114}$$

where σ is a supermatrix as introduced in section 2.4.2 and $\mathrm{D}\sigma$ is the invariant Berezinian superform.

2.6 Summary Chapter 2

We reviewed the historical development of random matrix theory. After a motivating example upon its application in principal component analysis and canonical correlation analysis, we summarized the theoretical background of the correlated Wishart and Jacobi ensemble.

In the ensemble approach, we replace the sample data matrix by an ensemble of model data matrices using the correlated Wishart model. Since many statistical properties of correlations in time series are carried by the eigenvalues of the correlation matrix, we focus in this thesis on its eigenvalue statistics. We reviewed how the joint eigenvalue distribution function is derived within the correlated Wishart model and summarized the challenges arising when eigenvalue statistics are considered.

We focus mainly on three aspects of the eigenvalue statistics, the statistics of the extremes, the eigenvalue density and the eigenvalue correlations. The statistics of the extremes are encoded in the distribution of the smallest and largest eigenvalue within the Wishart model. To study it, we present several representations of it and a related gap probability in terms of eigenvalue and matrix model averages. The eigenvalue density and correlations are studied via the k-point correlation function, which we expressed as an averaged k-fold product of ratios of characteristic polynomials.

For many of the next chapter's considerations, the method of supersymmetry plays an essential role. We reviewed important details of supermathematics, the superbosonization and the generalized Hubbard-Stratonovich transformation. As a first result of this thesis, we showed under modest assumptions on the probability distribution function how averaged products of determinants in the numerator as well as denominator in correlated Wishart models can be mapped to supermatrix models.

CHAPTER 3

Smallest Eigenvalue Statistics in Wishart-Laguerre Ensembles

This chapter is devoted to the computation of the smallest eigenvalue distribution and a related gap probability to find no eigenvalue within an interval $[0, s]$ in the uncorrelated real and real quaternion Wishart model, see section 2.3.3, also known as Wishart-Laguerre ensembles. For both models, these quantities still pose challenging difficulties, as averages over square roots of characteristic polynomials are involved. Because of these square roots standard random matrix techniques do not apply. The analysis of the real ensemble presented here is based on Refs. [3,5] and extends some of the aspects discussed therein.

Several attempts were made in the literature to study the smallest eigenvalue statistics in the real Wishart model. None of them obtained closed-form expressions in terms of integrable Pfaffian structures consisting of known functions. Only Ref. [129] studied the case of finite n, p. The author found a recursion relation for the distribution of the smallest eigenvalue, which, however, can be solved on a computer only. For $n - p = 0$ and $n - p = 2$ the microscopic limit of the smallest eigenvalue distribution was derived in Ref. [91] and in Ref. [130], respectively. Solely in the case of odd rectangularity $n-p = 2k+1$ and the complex Wishart-Laguerre model, which we will not consider here, closed-form expressions for the smallest eigenvalue distribution were obtained in Refs. [107,131] and Refs. [132,133], respectively.

For the real quaternion Wishart model, the distribution of the smallest eigenvalue was computed in terms of an infinite series in Ref. [134]. To the best of our knowledge, no Pfaffian structure for the smallest eigenvalue distribution and the gap probability are known in the literature.

The outline of this chapter is as follows: Section 3.1 is concerned with the gap probability and the distribution of the smallest eigenvalue for the real Wishart-Laguerre ensemble. In section 3.2, we study these quantities for the real quaternion ensemble. For clarity and comprehensibility we put long calculations into appendix A.

3.1 Real Wishart-Laguerre Ensemble

For the real Wishart-Laguerre ensemble, we consider the case of even rectangularity $\nu = n - p$ only, because for odd rectangularity square roots disappear such that standard techniques apply. Hence, in this case one obtains a closed-form expression in terms of an integrable Pfaffian [107, 131].

In section 3.1.1, we derive an integrable Pfaffian structure for the smallest eigenvalue distribution and a related gap probability. It is based on a class of non standard orthogonal polynomials, which we construct using bosonization techniques in section 3.1.2. Combining both results we compute in section 3.1.3 exact expressions as well as the hard-edge limit formulas of the gap probability. In section 3.1.4 we do the same for the smallest eigenvalue distribution. Since the analysis of both quantities in the hard edge limit is done for even p only, we show in section 3.1.5 using an alternative approach that the microscopic limit of the Pfaffian kernel is the same for even and odd p. We then extend this approach and sketch how to derive Pfaffian kernels without sums over orthogonal polynomials, establishing a kind of Christoffel-Darboux formula for the polynomials constructed in section 3.1.2. To illustrate our findings, we compare our analytic expressions to numerical simulations in section 3.1.6.

3.1.1 Integrable Pfaffian Structure

We show that the smallest eigenvalue distribution (2.41) as well as the related gap probability (2.42) have an integrable Pfaffian structure for even rectangularity $\nu = n - p$. Due to the absence of correlation, we can express the gap probability

$$\begin{aligned} E_p^{(1)}([0,s];0) &= \exp(-\frac{sp}{2}) K_{p\times n} \\ &\quad \times \int \mathrm{d}[X] \;\; |\Delta_p(X)| \prod_{i=1}^{p} (x_i + s)^{(\nu-1)/2} \exp\left(-x_i/2\right) \end{aligned} \tag{3.1}$$

$$= \exp(-\frac{sp}{2}) \left\langle \det{}^{(\nu-1)/2}\left(X + s\mathbf{1}_p\right) \right\rangle_{\frac{1}{2},0} \tag{3.2}$$

and the smallest eigenvalue distribution

$$\begin{aligned} \mathcal{P}_{\text{min}}^{(1)}(s) &= p s^{(\nu-1)/2} \exp(-\frac{sp}{2}) K_{p\times n} \\ &\quad \times \int \mathrm{d}[X] \;\; |\Delta_p(X)| \prod_{i=1}^{p} x_i (x_i + s)^{(\nu-1)/2} \exp\left(-x_i/2\right) \end{aligned} \tag{3.3}$$

$$= p s^{(\nu-1)/2} \exp(-\frac{sp}{2}) \left\langle \det{}^{(\nu-1)/2}\left(X + s\mathbf{1}_{p-1}\right) \right\rangle_{\frac{3}{2},0} \tag{3.4}$$

as an averaged characteristic polynomial. Here $K_{p\times n}$ is a normalization constant determined by $E_p^{(1)}([0,s];0) \to 1$ for $s \to \infty$ and

$$\langle f(X) \rangle_{i,s} = K_{p\times n} \int \mathrm{d}[X] \;\; |\Delta_p(X)|\, f(X) \prod_{j=1}^{p} w_i(x_j; s) \;, \tag{3.5}$$

where the weight w_i is yet to be defined. Using Selberg's integral [40], we compute the normalization constant and find

$$\frac{1}{K_{p\times n}} = 2^{pn/2}\prod_{j=0}^{p-1}\frac{\Gamma\left(\frac{j+3}{2}\right)\Gamma\left(\frac{j+2\alpha+1}{2}\right)}{\Gamma\left(3/2\right)} \ . \tag{3.6}$$

The main difficulty in computing the averages (3.1) and (3.3) is the half-integer power of the characteristic polynomial caused by an even ν. We lever out this difficulty, by introducing the s-dependent weight function

$$w_i(x;s) = \frac{x^i\exp\left(-x/2\right)}{\sqrt{x+s}} \tag{3.7}$$

such that the gap probability (3.1) reduces to

$$E_p^{(1)}([0,s];0) = K_{p\times n}e^{-pt/2}\int \mathrm{d}[X] \ \ |\Delta_p(X)|\det{}^{\alpha}\left(X+t\mathbf{1}_p\right)\prod_{i=1}^{p}w_0(x_i;s) \ , \tag{3.8}$$

where $\nu = n-p = 2\alpha$ with integer α. Analogously, we obtain an expression for the smallest eigenvalue distribution,

$$\mathcal{P}_{\min}^{(1)}(s) = ps^{(\nu-1)/2}\exp(-\frac{sp}{2})\left\langle\det{}^{\nu/2}\left(X+s\mathbf{1}_{p-1}\right)\right\rangle_{1,s} \tag{3.9}$$

with $w_1(x,s)$ as weight function. By moving the half-integer part of the characteristic polynomial into the weight, the averages over a characteristic polynomial to half-integer power (3.1) and (3.3) become averages of a characteristic polynomial to integer power (3.8) and (3.9). The average in Eq. (3.8) is brought to a more convenient form by introducing dummy source variables κ_i, $i = 1,\ldots,\alpha$, that will be set to $-s$ at the end of the calculation. For the gap probability (3.8) this leads to

$$\begin{aligned} E_p^{(1)}(t) &= K_{p\times n}e^{-ps/2}\prod_{i=1}^{\alpha}\lim_{\kappa_a\to -s}\left\langle\prod_{i=1}^{\alpha}\det\left(X-\kappa_a\mathbf{1}_p\right)\right\rangle_{0,s} \qquad (3.10)\\ &= K_{p\times n}e^{-pt/2}\prod_{i=1}^{\alpha}\lim_{\kappa_a\to -s}Z_p^{\alpha,w_0}(\kappa) \ , \qquad (3.11)\end{aligned}$$

whereas for the smallest eigenvalue distribution we arrive at

$$\mathcal{P}_{\min}^{(1)}(s) = ps^{(\nu-1)/2}\exp(-\frac{sp}{2})K_{p\times n}\prod_{i=1}^{\alpha}\lim_{\kappa_a\to -s}Z_{p-1}^{\alpha,w_1}(\kappa) \tag{3.12}$$

The α-point partition function $Z_M^{\alpha,w_i}(\kappa)$ introduced in Eq. (3.11) is of standard type and was solved several times in the literature, see Ref. [107,110,112] and references therein. Assuming $\alpha = 2m$ even, it becomes a Pfaffian expression with a $2m\times 2m$ dimensional kernel,

$$Z_M^{2m,w_i}(\kappa) = \frac{(-1)^{2m(2m-1)/2}M!}{(M+2m)!\Delta_{2m}(\kappa)}Z_{M+2m}^{0,w_i}(0)\mathrm{pf}\left[\mathcal{K}_{L+m}(\kappa_a,\kappa_b)\right] \ , \tag{3.13}$$

where $M = 2L + \chi$, with $\chi \in \{0, 1\}$ and $1 \leq a, b \leq 2m$. The quantity $Z_M^{0,w_i}(0)$ is an explicit function of s yet to be determined. The kernel of the Pfaffian (3.13) is given by

$$\mathcal{K}_l(\kappa_a, \kappa_b) = \sum_{j=0}^{l-1} \frac{R_{2j+1}(\kappa_a, s) R_{2j}(\kappa_b, s) - R_{2j+1}(\kappa_b, s) R_{2j}(\kappa_a, s)}{r_j(s)} , \tag{3.14}$$

where $R_i(y, s)$ are skew-orthogonal polynomials with respect to the weight function (3.7)

$$\langle R_{2j+1} | R_{2i} \rangle_{1,s} = r_j(s)\delta_{ij} , \quad \langle R_{2j} | R_{2i} \rangle_{1,s} = 0 = \langle R_{2j+1} | R_{2i+1} \rangle_{1,s} . \tag{3.15}$$

The $r_i(s)$ are the normalization constants of the skew-symmetric scalar products introduced as

$$\langle f | g \rangle_{1,s} = \int_0^\infty \mathrm{d}y \int_0^y \mathrm{d}x w(x; s) w(y; s) \left(f(x)g(y) - f(y)g(x) \right) . \tag{3.16}$$

If M is even the polynomials are uniquely determined up to a constant. The latter is fixed by choosing the polynomials to be monic, i.e. $R_i(x) = x^i + \ldots$ If $M = 2L+1$ is odd, i.e. $\chi = 1$, the polynomials are not uniquely determined by orthogonality. Instead of considering monic polynomials in the derivation of Eq. (3.13), we require them to satisfy

$$\int_0^\infty \mathrm{d}x R_j(x, s) w_i(s; x) = \delta_{j,2L} . \tag{3.17}$$

If the gap probability has odd $\alpha = 2m-1$, we introduce an additional dummy source variable κ_{2m} such that if we set $\kappa_{2m} \to \infty$ at the end we obtain our original expression. Then we use our results found earlier for even number of source variables and apply the limit $\kappa_{2m} \to \infty$ to it, i.e.

$$\lim_{\kappa_{2m} \to \infty} \left(\frac{-1}{\kappa_{2m}} \right)^M Z_M^{2m,w_i}(\kappa) , \tag{3.18}$$

before setting κ_i to $-s$, yielding a $2m \times 2m$-dimensional Pfaffian structure.

Eventually, we are left with the computation of skew-orthogonal polynomials with respect to the weight (3.7). These polynomials have coefficients analytic in s. Later, when setting all source variables κ_i to $-s$, the kernels become analytic functions in s rather than polynomials. In this way we circumvent the study of analytic functions in s by studying orthogonal polynomials in dummy source variables.

3.1.2 Orthogonal Polynomials

In the previous section, we reduced the calculation of the gap probability to the construction of a particular class of orthogonal polynomials with respect to the scalar product (3.16) and the weight function (3.7). Employing a recent result [40, 135], we express these polynomials as eigenvalue integrals

$$\begin{aligned} R_{j=2N}(y,s) &= K_{2N} \int \mathrm{d}[X] \;\; |\Delta_{2N}(X)| \det\left(X - y\mathbf{1}_{2N}\right) \prod_{j=1}^{2N} w_i(x_j;s) \\ &= K_{2N}\; R_{2N}^{(\eta)}(y)\Big|_{\eta=1} \end{aligned} \tag{3.19}$$

if the degree j of the polynomial is even and

$$\begin{aligned} R_{j=2N+1}(y,s) &= K_{2N+1} \int \mathrm{d}[X] \;\; |\Delta_{2N}(X)| \left(y + \mathrm{tr}X\right) \det\left(X - y\mathbf{1}_{2N}\right) \prod_{j=1}^{2N} w_i(x;s) \\ &= K_{2N+1} \left(y R_{2N}^{(\eta)}(y,s)\Big|_{\eta=1} - 2\,\frac{\partial}{\partial\eta} R_{2N}^{(\eta)}(y,s)\Big|_{\eta=1} \right) \;, \end{aligned} \tag{3.20}$$

if it is odd. The introduction of the "generating polynomial"

$$R_n^{(\eta)}(y,s) = \int \mathrm{d}[X] \;\; |\Delta_n(X)| \det\left(X - y\mathbf{1}_n\right) \frac{\exp\left(-\frac{\eta}{2}\mathrm{tr}X\right)}{\sqrt{\det\left(X + s\mathbf{1}_n\right)}}, \tag{3.21}$$

enables us to deduce both polynomials from the same calculation. We suppress the index i, indicating which weight function (3.7) has to be used. For the smallest eigenvalue distribution we take $i = 1$ and for the related gap probability $i = 0$. However, we construct the polynomials for all positive, integer values of i in a unified way. The normalization constants K_{2N} and K_{2N+1} of the orthogonal polynomials are fixed by requiring that these polynomials are monic, *i.e.*

$$\lim_{y\to\infty} \frac{1}{y^{2N}} R_{2N}(y,s) = 1 = \lim_{y\to\infty} \frac{1}{y^{2N+1}} R_{2N+1}(y,s) \;. \tag{3.22}$$

In appendix A.1, we construct an invariant two-by-two matrix model coupled to a scalar integral which is dual to the generating polynomial (3.21). We derive a complete set of orthogonal polynomials using Eqs. (3.19) and (3.20). It turns out that for even degree $j = 2N$, they are given by

$$\begin{aligned} \frac{R_{2N}^{a}(y,s)}{K_{2N} s^{(2i+1)/2}} = \frac{2N!}{(2N-a)!} \Bigg[&\mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{1+2i}{2}; \frac{s}{2}\right) L_{2N-a}^{(1+2i+a)}(y) \\ &+ \frac{2N+2i+1}{2} \mathrm{U}\left(\frac{2N+2i+3}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) L_{2N-a}^{(2i+a)}(y) \Bigg] \;, \end{aligned} \tag{3.23}$$

and for odd degree $j = 2N+1$ by

$$\begin{aligned}
&\frac{R^a_{2N+1}(y,s)}{K_{2N+1}s^{(2i+1)/2}} = (y + 2N(2N+2i+2) - 2i - 1)\, R^a_{2N}(y,s) \\
&+ aR^{a-1}_{2N}(y,s) - \frac{(2N)!}{(2N-a)!}\left(\mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{1+2i}{2}; \frac{s}{2}\right)\right. \\
&\times \left[2y(2N-a)L^{(2+a+2i)}_{2N-a-1}(y) + 2aL^{(1+a+2i)}_{2N-a}(y)\right] \\
&+ \frac{2N+2i+1}{2}\mathrm{U}\left(\frac{2N+2i+3}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) \\
&\times \left[sL^{(2i+1+a)}_{2N-a}(y) - 2y(2N-a)L^{(a+1+2i)}_{2N-a-1}(y) - 2aL^{(a+2i)}_{2N-a}(y)\right] \\
&\left.+ s\frac{2N+2i+1}{2}\frac{2N+2i+3}{2}\mathrm{U}\left(\frac{2N+2i+5}{2}, \frac{5+2i}{2}; \frac{s}{2}\right) L^{(2i+a)}_{2N-a}(y)\right) .
\end{aligned} \tag{3.24}$$

where $\mathrm{U}\,(a,b;z)$ is the confluent hypergeometric function and $L_N^{(b)}(y)$ is the Laguerre polynomial in monic normalization with $L_N^{(b)}(y) = 0$ for all $N < 0$, see Eq. (A.14). The superscript a in Eq. (3.23) and Eq. (3.24) denotes the ath derivative with respect to the "polynomial" argument y, which we compute for later purpose. Applying the normalization condition (3.22) to the even and odd polynomials Eqs. (3.23) and (3.24), determines K_{2N} and K_{2N+1} to be

$$K_{2N} = K_{2N+1} = \frac{1}{\sqrt{s}^{2i+1}\mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)} . \tag{3.25}$$

Besides the orthogonal polynomials, the Pfaffian kernel (3.14) depends on the normalization constants $r_m(s)$ of the scalar product, see also Eq. (3.15). To compute it, we make use of [40]

$$Z^{0,w_i}_{2m}(0) = \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_{2m}\, |\Delta_{2m}(X)| \prod_{j=1}^{2m} w_i(x_j,t) = (2m)! \prod_{j=0}^{m-1} r_j(s) \tag{3.26}$$

and find

$$r_m(s) = \frac{(2m)!}{(2m+2)!}\frac{Z^{0,w_i}_{2m+2}(0)}{Z^{0,w_i}_{2m}(0)} . \tag{3.27}$$

An analytic closed-form expression for the partition functions Z^{0,w_i}_{2m} derives from the generating polynomial, see Eq. (A.26). Inserting it into Eq. (3.27) yields

$$r_m(s) = 2(2m)!(2m+1+2i)!\frac{\mathrm{U}\left(\frac{2m+3+2i}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)}{\mathrm{U}\left(\frac{2m+1+2i}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)} . \tag{3.28}$$

If the number of eigenvalue integrals in Eq. (3.13) M is odd, we modify our polynomials to satisfy Eq. (3.17). As we have an odd number of polynomials, these monic polynomials are not uniquely determined by orthogonality [40,44]. Moreover, the highest degree polynomial is orthogonal to all other $M-1$ polynomials, see Eq. (3.15). Thus, we can add multiples of it to each polynomial without affecting the orthogonality as defined in Eq. (3.15). Introducing

$$\hat{g}_i(s) = \int_0^\infty \mathrm{d}x R_i(x,s) w_i(x,s) \;, \tag{3.29}$$

$$\hat{R}_i^a(y,s) = R_i(y,s) - g_i(s) R^a_{2L+2m}(y,s) \qquad \forall i = 1,\dots,p-2 \;, \tag{3.30}$$

$$\hat{R}^a_{2L+2m}(y,s) = \frac{1}{g_{2L+2m}(s)} R^a_{2L+2m}(y,s) \;, \tag{3.31}$$

where $R_i^a(y,s)$ are the polynomials as given by Eqs. (3.23) and (3.24), we find that the $\hat{R}_i^0(y,s)$ satisfy Eqs. (3.15) and (3.17).

3.1.3 Exact Expression and Microscopic Limit of $E_p^{(1)}$

The polynomials constructed in the previous sections, together with the results of section 3.1.1 yield a closed-form solution to Eq. (3.1), which we study in this section. We complete this section with an analysis of the gap probability in the microscopic limit. We begin our finite n, p analysis with the $p=2L$ even case and adjust it to the $p=2L+1$ odd case.

It is worth mentioning that in this section we are concerned only with the polynomials Eq. (3.23) and Eq. (3.24) obtain for $i=0$.

Even $p=2L$

Before we insert the analytic expressions obtained in section 3.1.2 into the generating function, we perform the limits (3.11). As this is straightforward, we only give the result. If $\alpha=2m$ is even, we obtain

$$\begin{aligned} E_p^{(1)}([0,s];0) = \mathcal{C}_{p,n} \mathrm{U}\left(\frac{p+2m+1}{2},\frac{3}{2};\frac{s}{2}\right) \exp\left(-ps/2\right)\sqrt{s} \\ \times \,\mathrm{pf}\left[\sum_{j=0}^{l-1} \frac{R^a_{2j+1}\left(-s,s\right) R^b_{2j}\left(-s,s\right) - (a \leftrightarrow b)}{r_j(s)}\right] \;, \end{aligned} \tag{3.32}$$

where $a,b=0,\dots,\alpha-1$, $l=L+m$ and the polynomials are given by Eqs. (3.23) and (3.24) with $i=0$. The superscript a indicates the ath derivative of the polynomials with respect to its first argument. The normalization constant is given by

$$\mathcal{C}_{p,n} = (-1)^{\alpha p} \prod_{i=0}^{\alpha-1} \frac{(2i)!}{i!} \frac{p! 2^{(2m-1)/2}}{(p+2m)!} \frac{\prod_{i=0}^{2m-1}(p+i+1)!}{\prod_{i=0}^{\alpha-1}(p+2i)!} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \;. \tag{3.33}$$

To find a Pfaffian structure for $\alpha = 2m-1$ odd, we introduce an additional dummy source variable κ_{2m}, apply our results for even α and perform eventually the limit (3.18). Together with the limit (3.11), we arrive at

$$E_p^{(1)}(t) = \mathcal{C}_{p,n} \exp(-ps/2)\sqrt{s}\,\mathrm{U}\left(\frac{p+2m+1}{2}, \frac{3}{2}; \frac{s}{2}\right)$$
$$\mathrm{pf}\left[\begin{array}{cc} \displaystyle\sum_{j=0}^{L+m-1} \frac{R^a_{2j+1}(-s,s)\,R^b_{2j}(-s,s) - (a \leftrightarrow b)}{r_j(s)} & -\dfrac{R^a_{2(L+m-1)}(-s,s)}{r_{L+m-1}(s)} \\ \dfrac{R^b_{2(L+m-1)}(-s,s)}{r_{L+m-1}(s)} & 0 \end{array}\right], \tag{3.34}$$

for the gap probability, where $1 \leq a, b, \leq 2m$. The polynomials used are given by Eqs. (3.23) and (3.24) with $i = 0$.

Odd $p = 2L+1$

We obtain a Pfaffian structure for odd $p = 2L+1$ if we put an additional requirement on the polynomials, *c.f.* Eq. (3.17). The polynomials for odd p are derived from those constructed above by shifting all polynomials, except the polynomial of highest degree, by appropriate scalar multiples of it , *c.f.* Eq. (3.30). Hence, the expression for the gap probability remains the same, apart from substituting the shifted polynomials $\hat{R}^a_j(-s,s)$ for $R^a_j(-s,s)$ in Eq. (3.32) and Eq. (3.34). In case of odd α, we replace the polynomial in the last row and column by

$$\frac{1}{g_{2L+2m}(s)} \sum_{j=0}^{L+m-1} \frac{R^{a-1}_{2j+1}(-s,s)\,g_{2j}(s) - g_{2j+1}(s)\,R^{a-1}_{2j}(-s,s)}{r_j(s)}, \tag{3.35}$$

where the polynomials are those constructed in section 3.1.2, as shown in Eq. (3.23) and Eq. (3.24) and $g_j(s)$ as introduced in Eq. (3.29).

The Microscopic Limit

We study the microscopic limit of the gap probability (3.1), *i.e.* we perform the limit $p \to \infty$, keeping $\nu = n - p$ fixed and focus on the scale $s = u/4p$,

$$\mathcal{E}^{(1)}(u) = \lim_{p\to\infty} E_p^{(1)}\left(\frac{u}{4p}\right). \tag{3.36}$$

We do the analysis for even p only and show in section 3.1.5 that both, the even and the odd p case, have the same microscopic limit.

The dimension of the matrix kernel in the Pfaffian determinants (3.32) and (3.34), depends solely on the rectangularity $\nu = n - p$. Accordingly, the Pfaffian structure of the gap probability survives in the microscopic limit. Hence, we analyze the asymptotic behavior of the normalization constant and the kernel only. To treat the latter, we replace the sum by an integral

$$\sum_{j=0}^{l-1} f_j \to \frac{p}{2}\int_0^1 \mathrm{d}x f_{x(p+2m-2)/2}, \tag{3.37}$$

where we use that $l = L + m = p/2 + m$. Since the contributions of $f_{x(p+2m-2)/2}$ to the integral for x smaller than $\mathcal{O}(1)$ are of measure zero, we require the large-p contribution to summand (3.37) only. It is given by the leading order of the polynomials (3.23) and (3.24) in p.

Substituting the asymptotic expressions for the polynomials and the scalar product normalization constants derived in appendix A.2 into the Pfaffian kernel (3.14) and replacing the sum by an integral as shown in Eq. (3.37), we obtain

$$p^{a+b+1}\Xi^{a,b}(u) = \frac{(-2p)^{a+b}p}{8}\int\limits_0^1 \frac{\mathrm{d}x}{\sqrt{ux}}\left(\frac{x}{u}\right)^{(a+b)/2}\left(2\,(b-a)\,\mathsf{I}_a\left(\sqrt{ux}\right)\mathsf{I}_b\left(\sqrt{ux}\right)\right.$$
$$\left. + \, (2b+1)\sqrt{ux}\mathsf{I}_{a+1}\left(\sqrt{ux}\right)\mathsf{I}_b\left(\sqrt{ux}\right) - (2a+1)\mathsf{I}_a\left(\sqrt{ux}\right)\mathsf{I}_{b+1}\left(\sqrt{ux}\right)\right)\ , \tag{3.38}$$

where $\mathsf{I}_b(x)$ is the modified Bessel function of first kind. The remaining integrals can be performed using

$$\int\limits_0^1 \mathrm{d}xy^c\mathsf{I}_\mu\left(y\sqrt{u}\right)\mathsf{I}_\nu\left(y\sqrt{u}\right) = 2^{-\sigma-1}u^\sigma\Gamma(\sigma+1)\Gamma\left(\frac{1}{2}(c+\sigma+1)\right)$$
$${}_3\tilde{F}_4\left(\frac{1}{2}(\sigma+1),\frac{1}{2}(\sigma+2),\frac{1}{2}(c+\sigma+1);\mu+1,\frac{1}{2}(c+\sigma+3),\nu+1,\sigma+1;u\right)\ , \tag{3.39}$$

where $\sigma = \mu + \nu$ and ${}_3\tilde{F}_4$ is the regularized counterpart of the hypergeometric function ${}_3F_4$, c.f. Ref. [136]. Although the representation of the kernel in terms of hypergeometric functions does not gain new insights, it drastically decreases the evaluation time when implementing our final expressions in a computer program.

To eventually take the microscopic limit of the gap probability, we derive an asymptotic expression of the overall normalization constant (3.33) in appendix A.3. We combine the results found there with the large-p expressions of the Pfaffian kernel in Eq. (3.47) and take the limit (3.36). For $\alpha = 2m$ even, it yields that the microscopic limit of the gap probability to find all eigenvalues within the uncorrelated Wishart model above a certain threshold u is given by

$$\mathcal{E}(u) = \frac{2^m\prod_{i=0}^{2m-1}(2i)!}{\sqrt{\pi}\prod_{i=0}^{2m-1}i!}\exp\left(-\frac{u}{8}\right)u^{1/4}\mathsf{K}_{1/2}\left(\sqrt{\frac{u}{4}}\right)\mathrm{pf}\left[\Xi^{(a,b)}(u)\right]\ , \tag{3.40}$$

where $0 \leq a, b \leq 2m-1$ and $\mathsf{K}_\mu(x)$ is the modified Bessel function of second kind. If $\alpha = 2m-1$ is odd, we have to study the microscopic limit of Eq. (3.34). We compute the leading order contribution in p of the ath derivative of the orthogonal polynomial in appendix A.3, see Eq. (A.30) with $x = 1$. Combining the asymptotics of the Γ-function coming from $r_{L+m}(s)$ in the denominator of the last row and column in Eq. (3.34) with the asymptotics of the overall normalization constant leads to

$$\begin{aligned}
\mathcal{E}(u) &= \frac{2^m \prod_{i=0}^{2m-2}(2i)!}{\sqrt{\pi}\prod_{i=0}^{2m-2} i!} \exp\left(-\frac{u}{8}\right) u^{1/4} \mathsf{K}_{1/2}\left(\sqrt{\frac{u}{4}}\right) \\
&\times \mathrm{pf}\left[\begin{array}{cc} \Xi^{(a,b)}(u) & -\frac{(-2)^a}{\sqrt{u}^a}\left(\mathsf{I}_{a+1}\left(\sqrt{u}\right)+\mathsf{I}_a\left(\sqrt{u}\right)\right) \\ \frac{(-2)^b}{4\sqrt{u}^a}\left(\mathsf{I}_{b+1}\left(\sqrt{u}\right)+\mathsf{I}_b\left(\sqrt{u}\right)\right) & 0 \end{array}\right],
\end{aligned} \tag{3.41}$$

where $0 \leq a, b \leq 2m-2$.

3.1.4 Exact Expression and Microscopic Limit of $\mathcal{P}_{\text{min}}^{(1)}$

We showed in section 2.3.3 that apart from a sign, differentiating the gap probability $E_p^{(1)}([0,s];0)$ with respect to s yields the smallest eigenvalue distribution, see Eq. (2.42). In general, a derivative of a Pfaffian does not lead to a Pfaffian in an obvious way. In section 3.1.1, we obtain an integrable Pfaffian structure for the smallest eigenvalue distribution. It depends only on the skew-orthogonal polynomials with respect to the weight $w_1(x;s)$. We constructed these polynomials in section 3.1.2. In the current section, we give a closed-form expression for the distribution of the smallest eigenvalue. First we consider the finite n, p case followed by the computation of its microscopic limit.

It is worthy to emphasize again that when considering the smallest eigenvalue distribution, we are concerned with the polynomials Eq. (3.23) and Eq. (3.24) obtained for $i = 1$ only.

Finite p, n Expression

The smallest eigenvalue distribution for $\alpha = 1$, which is known in the literature [129], can readily be derived from Eq. (A.16) by comparing Eq. (3.3) with Eq. (3.21) for $i = 1$, yielding

$$\begin{aligned}
\mathcal{P}_{\text{min}}^{(1)}(s) = \frac{(-1)^{p-1}e^{-ps/2}s^2}{2^{p+1}\Gamma(p/2)} &\left(L_{p-1}^{(3)}(-s)\mathsf{U}\left(\frac{p+2}{2},\frac{3}{2};\frac{s}{2}\right)\right. \\
&\left.+\frac{p+2}{2}L_{p-1}^{(2)}(-s)\mathsf{U}\left(\frac{p+4}{2},\frac{5}{2};\frac{s}{2}\right)\right).
\end{aligned} \tag{3.42}$$

We assume that $p = 2L+1$ is odd such that $p-1$ is even. For arbitrary α, we combine the results section 3.1.2 with $i = 1$ and those of section 3.1.1. This leads to closed-form expressions for the smallest eigenvalue distribution in terms of known functions. If $\alpha = 2m$ is even, it is given by

$$\begin{aligned}
\mathcal{P}_{\text{min}}^{(1)}(s) &= \tilde{\mathcal{C}}_{p,n} s^{\alpha+1} e^{-ps/2} \mathsf{U}\left(\frac{p+2m+2}{2},\frac{5}{2};\frac{s}{2}\right) \\
&\times \mathrm{pf}\left[\sum_{j=0}^{L+m-1} \frac{R_{2j+1}^a(-s,s)\,R_{2j}^b(-s,s)-(a\leftrightarrow b)}{\tilde{r}_j(s)}\right],
\end{aligned} \tag{3.43}$$

where $0 \leq a, b, \leq 2m-1$. The polynomials are given by Eqs. (3.23) and (3.24) with $i = 1$ and the normalization constant reads

$$\tilde{\mathcal{C}}_{p,n} = \frac{(-1)^{\alpha(p-1)} p! 2^{m-1}}{(p+2m-1)!} \prod_{j=0}^{\alpha-1} \frac{(2i)!}{i!} \frac{\Gamma\left(\frac{p+3}{2}\right)}{\sqrt{\pi}} \frac{\prod_{j=0}^{2m-2}(p+j+2)!}{\prod_{j=1}^{\alpha-1}(p+2j)!} \,. \tag{3.44}$$

As explained above Eq. (3.18), for odd $\alpha = 2m-1$, we introduce a dummy source variable κ_{2m} such that the limit $\kappa_{2m} \to \infty$ results in our original expression. Then we apply the results for even α and take the limit $\kappa_{2m} \to \infty$. This leads to a Pfaffian structure with a $2m \times 2m$-dimensional matrix kernel as in Eq. (3.43), where the last row and column is replaced by a polynomial. It reads

$$\begin{aligned} \mathcal{P}^{(1)}_{\min}(s) &= \tilde{\mathcal{C}}_{p,n} s^{\alpha+1} e^{-ps/2} \mathrm{U}\left(\frac{p+2m+2}{2}, \frac{5}{2}; \frac{s}{2}\right) \\ &\mathrm{pf}\begin{bmatrix} \displaystyle\sum_{j=0}^{L+m-1} \frac{R^a_{2j+1}(-s,s)\, R^b_{2j}(-s,s) - (a \leftrightarrow b)}{\tilde{r}_j(t)} & -\dfrac{R^a_{2L+2m-2}(-s,s)}{\tilde{r}_{L+m-1}(s)} \\ \dfrac{R^b_{2L+2m-2}(-s,s)}{\tilde{r}_{L+m-1}(s)} & 0 \end{bmatrix}, \end{aligned} \tag{3.45}$$

where $0 \leq a, b, \leq 2m-1$.

In section 3.1.3, we discussed for the gap probability only the differences between the even p and odd p case. The arguments given there hold in the present case. If $p-1$ is odd, *i.e.* p is even, we replace the polynomials R^a in Eq. (3.43) and Eq. (3.45) by the corresponding shifted polynomials $\widehat{R}^a$, see Eq. (3.30). Furthermore, if $\alpha = 2m-1$ is odd, we insert

$$\frac{1}{g_{2L+2m}(s)} \sum_{j=0}^{L+m-1} \frac{R^{a-1}_{2j+1}(-s,s)\, g_{2j}(s) - g_{2j+1}(s)\, R^{a-1}_{2j}(-s,s)}{r_j(s)} \,, \tag{3.46}$$

for the elements in the last row and column of the Pfaffian in Eq. (3.45).

The Microscopic Limit

We now turn to the microscopic limit of the smallest eigenvalue distribution. Analogous to the gap probability, we consider $p-1$ even only. By the arguments given in section 3.1.5, the microscopic limit turns out to be the same for even and odd values of $p-1$.

As explained in Eq. (3.37), we replace the sum in the Pfaffian determinant (3.43) by an integral and substitute the asymptotic expressions for the even as well as the odd polynomials and the scalar product normalization into it. We derive them as well as the resulting leading order contribution in p to the kernel, given by

$$
\begin{aligned}
p^{a+b+3}\Xi^{(a,b)}(u) &= (-1)^{a+b+1}p^{a+b+3}2^{a+b}\int\limits_0^1 \mathrm{d}x \frac{\left(\frac{x}{u}\right)^{(b+a+4)/2}}{\left(\frac{2}{\sqrt{ux}}+1\right)}x^2 \\
&\times \Bigg[4\left(a-b\right)\left(\frac{2}{\sqrt{ux}}+1\right)\mathsf{I}_{a+2}\left(\sqrt{ux}\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right) \\
&+\left(\left(\sqrt{ux}-10\right)-\frac{ux}{2+\sqrt{ux}}-2\left(2b-3\right)\right)\mathsf{I}_{a+3}\left(\sqrt{ux}\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right) \\
&-\left(\left(\sqrt{ux}-10\right)-\frac{ux}{2+\sqrt{ux}}-2\left(2a-3\right)\right)\mathsf{I}_{b+3}\left(\sqrt{ux}\right)\mathsf{I}_{a+2}\left(\sqrt{ux}\right)\Bigg],
\end{aligned}
\tag{3.47}
$$

in appendix A.4. The large-p behavior of the global normalization constant is computed in appendix A.5. We insert it into Eq. (3.43), replace the Pfaffian kernel by Eq. (3.47) and take the limit $p \to \infty$. This yields, for $\alpha = 2m$ even, the microscopic limit of the distribution of the smallest eigenvalue

$$
\wp^{(1)}_{\mathsf{min}}(u) = \frac{\prod_{j=0}^{\alpha-1}(2i)!\ u^{\alpha+1/4}}{2^{2m+3}\sqrt{\pi}\prod_{j=0}^{\alpha-1} i!}\mathsf{K}_{3/2}\left(\sqrt{\frac{u}{4}}\right)\exp\left(-\frac{u}{8}\right)\mathsf{pf}\left[\Xi^{(a,b)}(u)\right] , \tag{3.48}
$$

where $0 \leq a, b \leq \alpha - 1$. Whereas, for $\alpha = 2m - 1$ we substitute the asymptotic expressions obtained for the normalization constant (A.41) and the kernel (3.47) into Eq. (3.45) and find

$$
\begin{aligned}
\wp^{(1)}_{\mathsf{min}}(u) &= \frac{\prod_{j=0}^{\alpha-1}(2i)!}{2^{2m+1}\sqrt{\pi}\prod_{j=0}^{\alpha-1} i!}\mathsf{K}_{3/2}\left(\sqrt{\frac{u}{4}}\right)\exp\left(-\frac{u}{8}\right)u^{\alpha-3/4} \\
\mathsf{pf}&\begin{bmatrix} \Xi^{(a,b)}(u) & * \\ \frac{(-2)^b}{u^{b/2}}\left(\mathsf{I}_{b+2}\left(\sqrt{u}\right)+\frac{\sqrt{u}}{2+\sqrt{u}}\mathsf{I}_{b+3}\left(\sqrt{u}\right)\right) & 0 \end{bmatrix},
\end{aligned}
\tag{3.49}
$$

where $0 \leq a, b \leq \alpha - 1$ and $*$ is (-1) times the transposed of the last row in the Pfaffian determinant.

3.1.5 Alternative Approach

We present a second approach to study the α-point partition function Z_M^{α,w_i} as introduced in Eq. (3.11). To this end, we construct a four dimensional matrix model dual to the matrix kernel of the Pfaffian (3.14). We study the microscopic limit of it and show that it is the same for odd and even M. This completes the computation of the microscopic limit in section 3.1.3 and 3.1.4.

Dual Non-Invariant Matrix Model

Because the gap probability and the smallest eigenvalue distribution are special limits of the α-point partition function, the alternative approach is concerned with the latter only. It is given by

$$Z_M^{\alpha,w_i}(\kappa) = \int \mathrm{d}[X] \ \ |\Delta_M(X)| \prod_{i=1}^{\alpha} \det\left(X - \kappa_i \mathbf{1}_M\right) \prod_{j=1}^{M} w_i(x_j;t), \tag{3.50}$$

where we take $M = p-1$ and $M = p$ for the smallest eigenvalue distribution and the corresponding gap probability, respectively. For technical reasons, we assume that $\alpha = 2m$ is even and set $M = 2L + \chi$, where $\chi = \{0,1\}$. For α odd, we can derive a formula from the results of the even case, *c.f.* Eq. (3.18).

The alternative approach is based on a general treatment of eigenvalue integrals in Ref. [112], for earlier results see Refs. [40,44,107,110] and references therein. We sketch the important intermediate steps in appendix A.6 and find that the partition function (3.50) has the following Pfaffian structure

$$\begin{aligned} Z_M^{2m,w_i}(\kappa) = & \frac{(-1)^{2m(2m-1)/2} M!}{(M+2m)!\Delta_{2m}(\kappa)} Z_d^{0,w}(0) \\ & \times \mathrm{pf}\left[\frac{(-1)(\kappa_a - \kappa_b)(M+2m)! Z_{M+2m-2}^{2,w_i}(\kappa_a,\kappa_b)}{(M+2m-2)! Z_{M+2m}^{0,w_i}(0)}\right] . \end{aligned} \tag{3.51}$$

if $\alpha = 2m$ is even. Thus, we reduced the computation of the eigenvalue integrals in Eq. (3.50) to the calculation of

$$Z_{d-2}^{2,w_i}(\kappa_a,\kappa_b) = \int \mathrm{d}[X] \ \ |\Delta_{d-2}(X)| \prod_{i=a,b} \det\left(X - \kappa_i \mathbf{1}_{d-2}\right) \prod_{j=1}^{d-2} w_i(x_j;t) \ , \tag{3.52}$$

where $d = 2M + 2m$. Analogous to the construction of the orthogonal polynomial carried out in appendix A.1, the eigenvalue integral (3.52) is mapped to a four-by-four matrix model, coupled to a scalar integral. Because we average over two different determinants, we show in appendix A.7 that this leads to a non-invariant matrix model. It reads

$$\begin{aligned} Z_{d-2}^{2,w_i}(\kappa) = & K\sqrt{s}^{2i+1} \int_0^\infty \frac{\mathrm{d}x x^{(d+2i-3)/2}}{(1+x)^{d/2+1}} \int \frac{\mathrm{d}\mu(V)\ \det^{(d+2i-2)/2}\left(\mathbf{1}_4 + V\right)}{\det^{(d-2)/2} V} \\ & \det^{1/2}\left((x+1)\,\mathbf{1}_4 + V\right) \exp\left(-\frac{sx}{2} - \frac{1}{2}\mathrm{tr}\kappa V\right) \ , \end{aligned} \tag{3.53}$$

where $\kappa = \mathrm{diag}(\kappa_a,\kappa_a,\kappa_b,\kappa_b)$, $V \in \mathrm{CSE}(4) = \mathrm{U}(4)/\mathrm{USp}(4)$ and $\mathrm{d}\mu(V)$ is the Haar measure.

Asymptotics

We study the behavior of the matrix model (3.53) to show that its limiting expression is the same for even or odd M tending to infinity. This completes the discussion of the microscopic limit in section 3.1.3 and 3.1.4.

By expanding the determinant in the second line of Eq. (3.53), the x and the V-integral decouple such that we can analyze their asymptotic behavior separately. To perform this large-M limit, we derive the correct scaling of the parameters first. The quantity we are aiming to study is the smallest eigenvalue distribution (3.3) and the gap probability (3.1), where $M = p-1, p$, respectively. From the analysis of Ref. [91] it turns out that we have to study the kernel (3.53) on the scale $u = 4pt$. We use the same scaling for the microscopic limit of the distribution of the smallest eigenvalue and the gap probability. This immediately yields that we set $(\kappa_a, \kappa_b) = (\xi_a/4p, \xi_b/4p)$. On this scale Eq. (3.53) becomes

$$\begin{aligned} Z_{d-2}^{2,w_i}(\xi/4p) = K\sqrt{u}^{2i+1} \int_0^\infty \frac{\mathrm{d}x x^{(d+2i-3)/2}}{(1+x)^{d/2+1}} \int \frac{\mathrm{d}\mu(V)\ \exp\left(-\frac{ux}{8p} - \frac{1}{8p}\mathrm{tr}\xi V\right)}{\det^{(d-2)/2} V} \\ \times \det{}^{1/2}\left((x+1)\mathbf{1}_4 + V\right) \det{}^{(d+2i-2)/2}\left(\mathbf{1}_4 + V\right) \ , \end{aligned} \tag{3.54}$$

where $\xi = \mathrm{diag}(\xi_a, \xi_a, \xi_b, \xi_b)$. We rescale both integration variables x and V by $p/2$ such that the exponents become independent of p and arrive at

$$\begin{aligned} Z_{d-2}^{w_i}(\xi/4p) = K\sqrt{u}^{2i+1} \int_0^\infty \frac{\mathrm{d}x x^{(d+2i-3)/2}}{(2/p+x)^{d/2+1}} \int \frac{\mathrm{d}\mu(V)\ \exp\left(-\frac{ux}{16} - \frac{1}{16}\mathrm{tr}\xi V\right)}{\det^{(d-2)/2} V} \\ \times \det{}^{1/2}\left(\left(x+\frac{2}{p}\right)\mathbf{1}_4 + V\right) \det{}^{(d+2i-2)/2}\left(\frac{2}{p}\mathbf{1}_4 + V\right) \ . \end{aligned} \tag{3.55}$$

To take the $p \to \infty$ limit, it is sufficient to know the leading term in the large-p expansion of Eq. (3.55). We start the analysis with the x-dependent part of the integrand. Using $d = M + 2m$ and $M/p \to 1$ for $p \to \infty$, we obtain

$$\frac{x^{(d+2i-3)/2}}{(2/p+x)^{d/2+1}} = x^{-(5-2i)/2} \exp\left(-\frac{1}{x}\right) + \mathcal{O}(p^{-1}) \ . \tag{3.56}$$

In the V-dependent part of the integrand, a similar expansion in p leads to

$$\frac{\det^{(d+2i-2)/2}\left(\frac{2}{p}\mathbf{1}_4 + V\right)}{\det^{(d-2)/2} V} = \det{}^i V \exp\left(\mathrm{tr}V^\dagger\right) + \mathcal{O}(p^{-1}) \ , \tag{3.57}$$

and

$$\det{}^{1/2}\left(\left(x+\frac{2}{p}\right)\mathbf{1}_4 + V\right) = \det{}^{1/2}\left(x\mathbf{1}_4 + V\right) + \mathcal{O}(p^{-1}) \ . \tag{3.58}$$

In order to decouple the x and the V integral and we write the determinant on the right hand side of Eq. (3.58) as

$$\det{}^{1/2}\left(x\mathbf{1}_4 + V\right) = x^2 + \frac{x}{2}\mathrm{tr}V + \det{}^{1/2} V \ . \tag{3.59}$$

All powers of p, arising from a rescaling of the integration variables and the expansion of the integrands, are absorbed into the overall normalization constant K.

As the normalization constant does not depend on the matrix indices (a, b), its p dependence is taken out of the Pfaffian (3.51) and is absorbed into the global normalization constant of the partition function. The global normalization constant is determined by the requirement that the microscopic limit of the gap probability as well as the smallest eigenvalue distribution are normalized.

Substituting Eqs. (3.56), (3.57) and (3.58) into the two point function (3.54) yields

$$\begin{aligned} Z_{d-2}^{2,w_i}(\xi/4p) = K\sqrt{u}^{2i+1} \int_0^\infty \mathrm{d}x x^{-(5-2i)/2} \exp\left(-\frac{ux}{16} - \frac{1}{x}\right) \int \mathrm{d}\mu(V) \\ \left(x^2 + \frac{x}{2}\mathrm{tr}V + \det{}^{1/2}V\right) \det{}^i V \exp\left(\mathrm{tr}V^{-1} - \frac{1}{16}\mathrm{tr}\xi V\right) + \mathcal{O}(p^{-1}) \ . \end{aligned} \tag{3.60}$$

The leading order of the kernel (3.53) does not depend on p and is therefore finite in the microscopic limit and the same for even and odd p. Thus, the microscopic limit of the gap probability computed in section 3.1.3 and 3.1.4 is also valid for odd p, respectively, $p-1$, which completes the computation of the microscopic limit in section 3.1.3 and 3.1.4.

Further Consideration of the Alternative Approach

Although the dual model (3.53) comprises a non-invariant four-by-four matrix average, we are able to solve it completely. To do so, we diagonalize the matrix $V = U(r \otimes \mathbf{1}_2)U^\dagger$, where $U \in \mathrm{USp}(4)$ and $r = \mathrm{diag}(r_1, r_2)$ with $|r_i| = 1$, yielding an eigenvalue integral and an average over $\mathrm{USp}(4)$. The latter is the first non-trivial example of the unitary-symplectic Itzykson-Zuber integral, see Eq. (2.21). Because closed-form expressions in terms of known functions exist, Ref. [67], we are left with the eigenvalue integral. The remaining integrals lead to a linear combinations of Laguerre polynomials dressed with confluent hypergeometric functions. A detailed discussion is given in appendix A.8, leading to

$$\begin{aligned} Z_{d-2}^{2,w_i}(\kappa) = \frac{K\sqrt{s}^{2i+1}}{(\kappa_a-\kappa_b)^3}\left(1 + \frac{(\kappa_a-\kappa_b)}{2}(\partial_{\kappa_a} - \partial_{\kappa_b})\right)(\partial_{\kappa_a} - \partial_{\kappa_a}) \\ \left[\mathrm{U}\left(\frac{d+2i-1}{2}, \frac{2i+3}{2}; \frac{s}{2}\right) + 4\mathrm{U}\left(\frac{d+2i-1}{2}, \frac{2i-1}{2}; \frac{s}{2}\right)\partial_{\kappa_a}\partial_{\kappa_b}\right. \\ \left. + 2\mathrm{U}\left(\frac{d+2i-1}{2}, \frac{2i+1}{2}; \frac{s}{2}\right)(\partial_{\kappa_a} + \partial_{\kappa_b})\right] L_d^{(2i-2)}(\kappa_a/2) L_d^{(2i-2)}(\kappa_b/2) \ . \end{aligned} \tag{3.61}$$

where the normalization constant is given by

$$K = \frac{2^{(d-2i+2)/2+2(d-2)}}{d(d-1)} \prod_{j=0}^{d-3}(j+1)! \prod_{k=0}^{i-1} \frac{(d+2i-1)!}{(2k+1)!} \tag{3.62}$$

Hence, instead of computing the α-point partition function (3.50) in terms of a Pfaffian with a kernel given by a sum over orthogonal polynomials as in Eq. (3.14), we obtain a rather compact matrix kernel. An evaluation of it is much faster compared to earlier results in section 3.1.3 and 3.1.4.

Moreover, expression (3.61) holds for both even and odd M. To derive an expression for the smallest eigenvalue distribution and the corresponding gap probability we set $i = 1$ and $i = 0$, respectively and perform the limit explained in Eqs. (3.12) and (3.11). To compute both quantities for odd α, we use the arguments given above Eq. (3.18).

It is worth remarking that Eq. (3.61) is a kind of Christoffel-Darboux formula for the skew-symmetric sum in Eq. (3.14). The skew-symmetric polynomial orthogonal with respect to the ordinary Laguerre weight is a special limit in s of those constructed in section 3.1.2. Thus, the Christoffel-Darboux kind of relation holds for an even broader class. This suggest that such an identity might exist for the skew-symmetric polynomials in general.

3.1.6 Numerical Simulations

Although our results are based on exact calculations, we compare them to numerical simulations for the purpose of illustrating and to confirm the correctness of our expressions. We start with the comparison of the exact results and then turn to the asymptotic formulas.

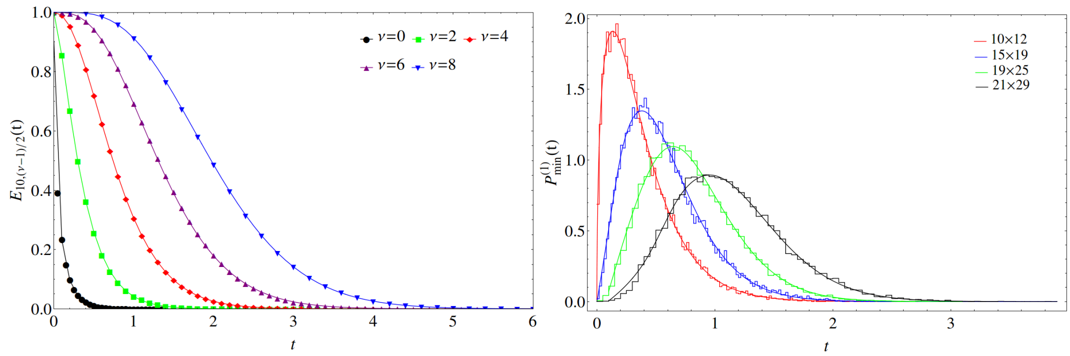


Figure 3.1: Left: Exact results (lines) compared to numerical simulations (symbols) of the gap probability for 10×10 Wishart correlation matrices for different rectangularities $\nu = 0, 2, 4, 6, 8$. Right: Exact results (lines) compared to numerical simulations (histograms) of the smallest eigenvalue distribution for Wishart correlation matrices consisting of rectangular matrices with different sizes.

We implement our formulas for the gap probability (3.32) and (3.34) as well as for the smallest eigenvalue distribution (3.43) and (3.45) in Mathematica [137]. We generate 10 000 samples of $10 \times (10 + \nu)$ dimensional real rectangular matrices, compute $WW^\dagger$ and compare the probability to find no eigenvalue in an interval $[0, t]$ within this sample to the exact expressions (3.32) and (3.34) for $\nu = 0, 2, 4, 6, 8$, see Fig. 3.1. Analogously, we generate 20 000 samples of real correlated Wishart matrices, compute numerically the distribution of the smallest eigenvalue and compare it to Eqs. (3.43) and (3.45). We carry this out for matrices W of size 10×12, 15×19, 15×19 and 21×29, see Fig. 3.1.

In section 4.1.4, we discuss a universality of the smallest eigenvalue distribution

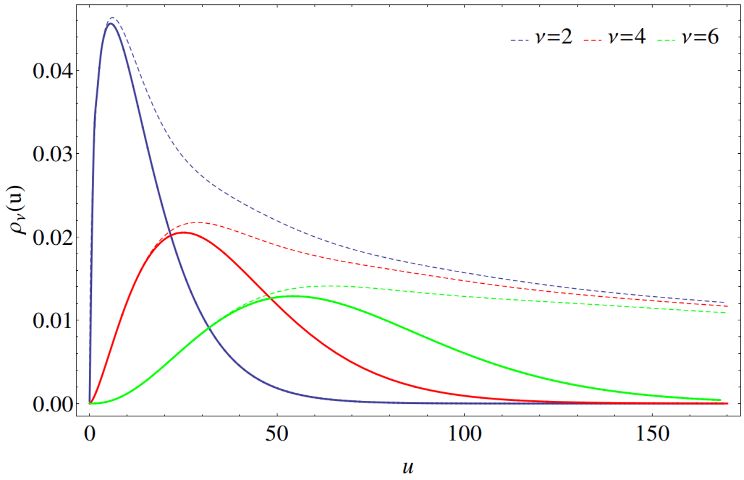


Figure 3.2: The microscopic density of states (dashed) compared to the smallest eigenvalue distribution (lines).

within the real and complex correlated Wishart model. It relates the statistics of the smallest eigenvalue in the correlated and uncorrelated Wishart ensemble. But, the analysis done there does not provide an analytic closed-form expression for the distribution of the smallest eigenvalue. Therefore, we postpone a comparison of the microscopic limiting distribution of the smallest eigenvalue within the real Wishart model to Monte-Carlo simulations to section 4.1.5, where correlations are included.

Since the contributions of the level density close to zero are caused by the statistics of the smallest eigenvalue, it is reasonable to compare analytic results for both quantities. We do this in the microscopic limit, where the contribution of the low-lying eigenvalues are crucial for comparisons with lattice QCD results. A closed-form expression for the former was found in Ref. [57, 138] and is given by

$$\rho_\nu(u) = \frac{1}{4}\left(J_\nu(\sqrt{u})^2 - J_{\nu-1}(\sqrt{u})J_{\nu+1}(\sqrt{u})\right) + \frac{1}{4\sqrt{u}}J_\nu(\sqrt{u})\left(1 - \int\limits_0^{\sqrt{u}} ds J_\nu(s)\right) . \tag{3.63}$$

In Fig. 3.2, we compare for $\nu = 2\alpha = 2, 4, 6$ the analytic expressions, showing a very good agreement for small values of u.

3.2 Real Quaternion Wishart-Laguerre Ensemble

For the real quaternion Wishart-Laguerre ensemble the distribution of the smallest eigenvalue was computed in Ref. [134] and is given in terms of a power series expansion. It is unclear whether it provides an integrable Pfaffian structure in terms of known functions.

We adapt the ideas of the previous section to the statistics of the real quaternion Wishart model and show that it indeed has a Pfaffian structure. Analogous to the real Wishart-Laguerre ensemble with even rectangularity, it depends on a new class of skew-orthogonal polynomials, yet to be computed.

In section 3.2.1, we show that the smallest eigenvalue distribution as well as the related gap probability have a Pfaffian structure. Similar to section 3.1.2, we develop an ordinary dual 3×3 matrix model representing the skew-orthogonal polynomials in section 3.2.2.

3.2.1 Integrable Pfaffian Structure

We study the smallest eigenvalue distribution using its corresponding gap probability, see section 2.3.3. The joint eigenvalue distribution function appearing in Eq. (2.37) is given by Eq. (2.20) with $\Lambda = \mathbf{1}_p$. Thus, the unitary-symplectic Itzykson-Zuber integral is trivial such that the gap probability (2.37) becomes

$$E_p^{(4)}([0,s];0) = K_{p\times n} \int_s^\infty \mathrm{d}[X] \ |\Delta_p(X)|^4 \prod_{i=1}^p x_i^{2\nu+1} \exp(-4x_i) \tag{3.64}$$

$$= K_{p\times n}\, e^{-4sp} \int_0^\infty \mathrm{d}[X] \ |\Delta_p(X)|^4 \prod_{i=1}^p (x_i+s)^{2\nu+1} \exp(-4x_i) \ . \tag{3.65}$$

The normalization constant $K_{p\times n}$ is determined by the requirement $E_p^{(4)}([0,s];0) \to 1$ for $s \to 0$. Employing this limit to Eq. (3.64), the resulting eigenvalue integral is a Selberg integral [40], yielding

$$\begin{aligned} K_{p\times n}^{-1} &= \int_0^\infty \mathrm{d}[X] \ |\Delta_p(X)|^4 \prod_{i=1}^p x_i^{2\nu+1} \exp(-4x_i) \\ &= \left(\frac{1}{4}\right)^{2pn} \prod_{j=0}^{p-1} \frac{\Gamma(3+2j)\,\Gamma(2\nu+2+2j)}{2} \ . \end{aligned} \tag{3.66}$$

To show that the gap probability (3.65) possesses a Pfaffian structure, we introduce a new weight function

$$w(x;s) = (x+s)\exp(-x) \ . \tag{3.67}$$

With the aid of which the gap probability (3.65) is written as an average of a product of characteristic polynomials to an even power,

$$\begin{aligned} E_p^{(4)}([0,s];0) = {} & K_{p\times n}\, e^{-4sp} \prod_{a=1}^{\nu} \lim_{\kappa_a \to -s} \int_0^\infty \mathrm{d}[X] \ |\Delta_p(X)|^4 \\ & \times \prod_{i=1}^p w(x_i;s) \prod_{a=1}^{\nu} \det{}^2(X-\kappa_a \mathbf{1}_p) \end{aligned} \tag{3.68}$$

$$= K_{p\times n}\, e^{-4sp} \prod_{a=1}^{\nu} \lim_{\kappa_a \to -s} Z_p^{\nu/0}(\kappa) \ , \tag{3.69}$$

where $\kappa = \mathrm{diag}(\kappa_1, \ldots, \kappa_\nu)$. In the real quaternion case, these averages were considered in Refs. [110,112] and references therein. In appendix A.9, we adapt the analysis of Ref. [112] and show that the partition function $Z_p^{\nu/0}(\kappa)$ as introduced in Eq. (3.69) indeed possesses a Pfaffian structure. For even $\nu = 2m$ it is given by

$$Z_p^{\nu/0}(\kappa) = \frac{(-1)^{(2p+2m)(2p+2m-1)/2+p(p-1)/2+2m(2m-1)/2} p!}{\Delta_{2m}(\kappa_1, \ldots, \kappa_{2m})} \mathrm{pf} M_{2p+2m}(s) \times \mathrm{pf}\left[\sum_{j=0}^{p+m-1} \frac{Q_{2j+1}(\kappa_a)Q_{2j}(\kappa_b) - Q_{2j}(\kappa_a)Q_{2j+1}(\kappa_b)}{r_j(s)}\right] , \tag{3.70}$$

where $1 \leq a, b \leq 2m$ and

$$\mathrm{pf} M_{2p+2m}(s) = \frac{(-1)^{(2p+2m)(2p+2m-1)/2+(p+m)(p+m-1)/2}}{(p+m)!} Z_{p+m}^{0/0}(0) . \tag{3.71}$$

The partition function $Z_{p+m}^{0/0}(0)$ on the right hand side of Eq. (3.71) is a Selberg integral and can be read off from Eq. (3.66) by setting $\nu = 0$ and $p \mapsto p + m$. The polynomials $Q_j(y)$ are skew-orthogonal with respect to the weight (3.67), *i.e.*

$$\langle Q_{2j+1} | Q_{2i} \rangle_{4,s} = r_i(s)\delta_{ij} , \quad \langle Q_{2j+1} | Q_{2i+1} \rangle_{4,s} = \langle Q_{2j} | Q_{2i} \rangle_{4,s} = 0 , \tag{3.72}$$

where the scalar product is given by

$$\langle f, g \rangle_{4,s} = \int_0^\infty \mathrm{d}x w(x;s) \left(f(x)g'(x) - f'(x)g(x)\right) . \tag{3.73}$$

For odd $\nu = 2m - 1$, we introduce a dummy variable κ_{2m}, use the results for even ν and take the limit $\kappa_{2m} \to \infty$ to obtain the following Pfaffian structure

$$Z_p^{\nu/0}(\kappa) = \frac{(-1)^{(2p+2m)(2p+2m-1)/2+p(p-1)/2+2m(2m-1)/2} p!}{\Delta_{2m-1}(\kappa_1, \ldots, \kappa_{2m-1})} \frac{\mathrm{pf} M_{2p+2m}(s)}{r_{p+m-1}(s)} \times \mathrm{pf}\begin{bmatrix} \sum_{j=0}^{p+m-1} \frac{Q_{2j+1}(\kappa_a)Q_{2j}(\kappa_b) - Q_{2j}(\kappa_a)Q_{2j+1}(\kappa_b)}{r_j(s)} & -Q_{2p+2m-1}(\kappa_b) \\ Q_{2p+2m-1}(\kappa_a) & 0 \end{bmatrix} , \tag{3.74}$$

where $1 \leq a, b \leq 2m - 1$. Thus, we reduced the computation of the partition function (3.69) to the calculation of the skew-orthogonal polynomials $Q_j(x)$ and their scalar product normalizations $r_i(s)$. To derive from Eqs. (3.70) and (3.74) the gap probability, we take the limit $\kappa_a \to -s$ for all a, leading to

$$E_p^{(4)}([0,s];0) = K_{p\times n}\, e^{-4sp} \frac{(-1)^{(p+m)(p+m-1)/2+p(p-1)/2} p!}{(p+m)! \prod_{j=1}^{2m-1} j!} \times Z_{p+m}^{0/0}(0) \mathrm{pf}\left[\sum_{j=0}^{p+m-1} \frac{Q_{2j+1}^a(-s)Q_{2j}^b(-s) - (a \leftrightarrow b)}{r_j(s)}\right] , \tag{3.75}$$

if ν is even and to

$$E_p^{(4)}([0,s];0) = K_{p\times n}\ e^{-4sp}\frac{(-1)^{(p+m)(p+m-1)/2+p(p-1)/2}p!}{(p+m)!\prod_{j=1}^{2m-2}j!}\frac{Z_{p+m}^{0/0}(0)}{r_{p+m-1}(s)}$$
$$\text{pf}\begin{bmatrix}\displaystyle\sum_{j=0}^{p+m-1}\frac{Q_{2j+1}^a(-s)Q_{2j}^b(-s)-(a\leftrightarrow b)}{r_j(s)} & -Q_{2p+2m-1}^b(-s)\\ Q_{2p+2m-1}^a(-s) & 0\end{bmatrix}\ , \tag{3.76}$$

if ν is odd, where $0\le a,b\le \nu-1$. Here $Q_j^a(x)$ is the ath derivative of the skew-orthogonal polynomials $Q_j(x)$ with respect to x.

We apply a similar analysis to the smallest eigenvalue distribution. To see that it has an integrable Pfaffian structure, we take the derivative with respect to s of the gap probability (3.64) and perform the shift $X\to X+s\mathbf{1}_p$, yielding

$$\begin{aligned}\mathcal{P}_{\text{min}}^{(4)}(s) &= -\frac{\mathrm{d}}{\mathrm{d}s}E_p^{(4)}([0,s];0) = K_{p\times n}\exp\left(-4sp\right)s^{2\nu+1}\\ &\times\int_0^\infty \mathrm{d}[X]\ \ |\Delta_{p-1}(X)|^4\prod_{i=1}^{p-1}(x_i+s)^{2\nu}\tilde{w}(x_i;s)\ ,\end{aligned} \tag{3.77}$$

where we introduced another weight function, namely

$$\tilde{w}(x;s) = (x+s)x^4\exp\left(-4x\right)\ . \tag{3.78}$$

The analysis done in appendix A.9 for the gap probability (3.64) can be adapted to Eq. (3.77) if we simply choose the polynomials $Q_j(x)$ to be skew-orthogonal with respect to the weight (3.78). This shows that the smallest eigenvalue distribution has an integrable Pfaffian structure as well and to compute it we calculate the polynomials orthogonal with respect to the weight (3.78) only.

3.2.2 Constructing the Polynomials for $w(x;s)$

The construction of the polynomials combines results of the theory of skew-symmetric polynomials and the method of Grassmann variables. It relies on an eigenvalue integral representation [40, 135],

$$Q_{2j}(y) = K_{2j,Q}\int \mathrm{d}[X]\ \ |\Delta_j(X)|^4\prod_{i=1}^{j}(x_i-y)^2 w(x;s) \tag{3.79}$$

for the polynomials of even degree and

$$Q_{2j+1}(y) = K_{2j+1,Q}\int \mathrm{d}[X]\ \ |\Delta_j(X)|^4\left(x+2\sum_{i=1}^{j}y_i\right)\prod_{i=1}^{j}(x_i-y)^2 w(x;s) \tag{3.80}$$

for polynomials of odd degree. The normalization constant $K_{j,Q}$ introduced in Eqs. (3.79) and (3.80) are choosen such that the polynomials are monic, meaning that $Q_j(x)=x^j+\ldots$ Similar to the construction of the polynomials in section

3.1.2, we derive the skew-orthogonal polynomials of even and odd degree from a generating polynomial

$$
\begin{aligned}
Q_{2j}(y) &= K_{2j,Q} Q_{2j}^{(1)}(y) \ , \\
Q_{2j}(y) &= K_{2j+1,Q} \left(y - \frac{1}{2}\partial_\eta \right)\Big|_{\eta=1} Q_{2j}^{(\eta)}(y) \ ,
\end{aligned}
\tag{3.81}
$$

where we introduced the generating polynomial

$$
Q_{2n}^{(\eta)}(y) = \int \mathrm{d}[X] \ \ |\Delta_n(X)|^4 \prod_{i=1}^{n} (x_i - y)^2 \exp\left(-4\eta x_i\right)(x_i + s) \ . \tag{3.82}
$$

The polynomial (3.82) is computed using Berezinian integrals and bosonization. To this end, we construct an underlying full matrix model. It is fixed by the requirement that diagonalization leads to Eq. (3.82). As in the previous section we take $\bar{W}$ to be a $n \times m$ matrix with real quaterntion entries such that $n \leq m$. Diagonalization of $\bar{W}\bar{W}^\dagger = U(X \otimes 1_2)U^\dagger$, where $X = \mathrm{diag}(x_1, \ldots, x_n)$ are the distinct eigenvalues and $U \in \mathrm{USp}(2n)$, induces a decomposition of the volume form,

$$
\mathrm{d}[\bar{W}] \ \sim |\Delta_n(X)|^4 \prod_{i=1}^{n} x_i^{2(n-m)+1} \mathrm{d}[X] \ \mathrm{d}\mu(U) \ , \tag{3.83}
$$

where $\mathrm{d}\mu(U)$ is the Haar measure. From Eq. (3.83) it is obvious that it is not possible to choose n and m such that the product over the monomials to a power $2(n-m)+1$ disappears. Hence, chosing the underlying full matrix model to be of real quaternion Wishart type, leads to a determinant in the denominator. This determinant is caused by the fact that we have to introduce a monomial factor as shown in Eq. (3.83) into the integrand (3.82).

Instead of a real quaternion Wishart model, we study a different symmetry class. We use the ensemble of Bogoliubov-deGennes Hamiltonians with time reversal symmetry. In an appropriate basis these are of the form

$$
\mathcal{H} = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}, \tag{3.84}
$$

where B is a $2n \times 2n$ dimensional complex, antisymmetric matrix. It is worth mentioning, that the physical context where this matrix model was originally introduced does not play a role. We introduce them for technical purposes only.

If we diagonalize the Hermitian matrix $BB^\dagger = \hat{U}(X \otimes 1_2)\hat{U}^\dagger$, where $\hat{U} \in \mathrm{U}(2n)/\left[\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SO}(2)\right]^n$ and $X = \mathrm{diag}(x_1, \ldots, x_n)$ is the matrix of distinct eigenvalues, the volume form becomes

$$
\mathrm{d}[B] \ \sim |\Delta_n(X)|^4 \ \mathrm{d}[X] \ \mathrm{d}\mu(\hat{U}) \ . \tag{3.85}
$$

Comparing the volume element (3.85) with the integrand of the generating orthogonal polynomial (3.82), we observe that this matrix ensemble is an ideal candidate to be an underlying model. Because, when going from the eigenvalue space in Eq. (3.82) to a B integral the overall structure of the integrand is not changed. Thus, we arrive at

$$\begin{aligned} Q_{2n}^{(\eta)}(y) = K \int \mathrm{d}[B] \ \mathrm{pf}^2 \left[\begin{array}{cc} B^\dagger & \sqrt{y}\mathbf{1}_{2n} \\ -\sqrt{y}\mathbf{1}_{2n} & -B \end{array} \right] \\ \times \mathrm{pf} \left[\begin{array}{cc} B^\dagger & \sqrt{t}\mathbf{1}_{2n} \\ -\sqrt{t}\mathbf{1}_{2n} & B \end{array} \right] \exp\left(-2\eta \mathrm{tr} BB^\dagger\right) \end{aligned} \tag{3.86}$$

To see that Eq. (3.86) and Eq. (3.82) coincide, we write the Pfaffian as an ordinary determinant using

$$\mathrm{pf} \left[\begin{array}{cc} B^\dagger & \sqrt{t}\mathbf{1}_{2n} \\ -\sqrt{t}\mathbf{1}_{2n} & B \end{array} \right] = \sqrt{\det(BB^\dagger + t\mathbf{1}_{2n})} \ . \tag{3.87}$$

The reason to use Pfaffians instead of square roots of determinants is that since $BB^\dagger$ is not symmetric; a determinant to a half-integer power of it can not be written as a Gaussian integral. However, we express the Pfaffians as Gaussian integrals over real Grassmann variables, see also Eq. (2.67), and map the average (3.86) to a dual non-invariant three-by-three matrix model,

$$Q_{2n}^{(\eta)}(y) = K \int \mathrm{d}[O] \ \det{}^{-2n-1} O \exp\left(2\mathrm{tr}\lambda O - \frac{1}{4\eta^2}\mathrm{tr} O^2\right) \ , \tag{3.88}$$

where $O \in \mathrm{COE}(3) = \mathrm{U}(3)/\mathrm{O}(3)$. A detailed construction of Eq. (3.88) is given in appendix A.10. Because of the $\mathrm{tr}\lambda O$ in the exponent, the COE average (3.88) is not invariant under the action of $O(3)$. Thus, diagonalizing $O = vrv^T$, where $v \in \mathrm{O}(3)$, $r = \mathrm{diag}(r_1, r_2, r_3)$ and $r_i \in \mathbb{C}$ with $|r_i| = 1$, leads to an orthogonal Itzykson-Zuber integral in three dimensions. Taking into account the structure of λ, this group integral reduces to a O(3)/O(2)-coset integral which isn't known in the literature.

Hence, we observe that because of the different symmetries, we are not able to simply adapt the analysis of the previous section for the real Wishart-Laguerre ensemble to the present case. The computation of the orthogonal polynomials is left for future work. We close this section by emphasizing that calculating the polynomial $Q_{2n}^{(\eta)}(y)$ not only solves the eigenvalue integral for the gap probability but also the non-trivial coset integral (3.88).

3.3 Summary Chapter 3

We considered the distribution of the smallest eigenvalue and a related gap probability in the uncorrelated real and real quaternion Wishart model. We showed that both quantities have an unexpected integrable Pfaffian structure.

For the real Wishart-Laguerre model, both the distribution of the smallest eigenvalue as well as the related gap probability are given in terms of an average of a characteristic polynomial to a half-integer power. We were able to circumvent this difficulty by shifting the half-integer power into the weight of the average. The resulting averaged characteristic polynomial to integer power was approached using standard random matrix theory techniques, yielding an integrable Pfaffian structure of dimension $\nu \times \nu$ if $\nu/2 = \alpha$ is even and $(\nu+1) \times (\nu+1)$ if it is odd. As matrix kernel of the Pfaffian, we obtained a skew-symmetric sum over polynomials orthogonal with respect to the new weight. Accordingly, we shifted the difficulty to the computation of the non-standard polynomials. The polynomials were constructed employing bosonization, yielding a linear combination of ordinary Laguerre polynomials with coefficients given by Tricomi's confluent hypergeometric function. The generality of this approach facilitates the derivation of all other important quantities such as the overall and the scalar product normalization constants. This leads to an exact expression for the smallest eigenvalue distribution as well as the gap probability.

The formulas we obtained are eminently suitable for the microscopic limit, because the size of the Pfaffian does not change. Thus, we reduced the calculation of the limiting distribution to the microscopic limit of the kernels. We replaced the sums by integrals and the polynomials by their asymptotic expressions and arrived at a closed-form expression for both quantities in terms of known functions.

We confirmed our findings with numerical simulations. We compare the gap probability as well as the smallest eigenvalue distribution for different dimensions to Monte-Carlo simulations and obtained a perfect agreement. In the microscopic limit, we compared the smallest eigenvalue distribution to the level density to establish that the contributions of the density close to zero are approximately generated by the statistics of the smallest eigenvalue.

For the smallest eigenvalue distribution and the related gap probability within the real quaternion Wishart model, we were able to partially adapt the analysis done for the real Wishart ensemble. We showed that both quantities provide an integrable Pfaffian structure by shifting a characteristic polynomial to a half-integer power into the weight. Employing standard techniques, we resulted in the desired Pfaffian expression. The kernel of the Pfaffian is given in terms of a skew-symmetric sum over polynomials orthogonal with respect to the new weight. To construct this non-standard class of polynomials, we used an average over the Bogoliubov-deGennes Hamiltonians with time reversal symmetry and mapped it using bosonization to a non-invariant three-by-three matrix model. Because of a symmetry breaking term, diagonalization of this matrix model led to a highly non-trivial group integral.

Hence, this representation did not led to a closed-form expression in terms of known functions. Nonetheless, we were able to gain new structural insights into the statistics of the smallest eigenvalue and reduced the complex integral in Eq. (3.65) to the computation of an integral over a three-by-three matrix, depending on three parameters, the dimension n, the polynomial argument y and the threshold s. Moreover, we obtained a relation between the smallest eigenvalue distribution within the real-quaternion Wishart model and an average over the Bogoliubov-deGennes Hamiltonians with time reversal symmetry.

CHAPTER 4

Supersymmetry in Correlated Ensembles

This chapter is devoted to the analysis of different statistical quantities within the correlated Wishart model using the method of supersymmetry. We study the statistics of the smallest as well as the largest eigenvalue and show that related gap probabilities provide a dual invariant matrix model. We derive a closed-form expression for the smallest eigenvalue distribution as well as the related gap probability. For the latter, we find, for the first time, a Pfaffian structure in the case of a real correlated Wishart model. Moreover, we show that in the microscopic limit the smallest eigenvalue statistics are universal and independent of the empirical eigenvalues. For the largest eigenvalue, we recover known results for the complex ensemble.

Apart from the statistics of extremes, we analyze for large matrix size the k-point function within the real Wishart model. We obtain an unexpected asymptotic relation between the bulk statistics of real Wishart ensembles with non-degenerate and with degenerate empirical eigenvalue spectra.

Applying new techniques invented recently [139,140], we compute the level density for the real and the complex correlated Jacobi ensemble. For the complex case, we derive a closed-form expression and for the real ensemble, we reduce the calculation to a non-trivial twofold integral.

In section 4.1, we analyze the smallest eigenvalue statistics within the real and the complex correlated Wishart model. We continue the discussion of the extreme eigenvalues in section 4.2 by studying the probability to find the largest eigenvalue of a Wishart correlation matrix within the interval $[0, t]$. In section 4.3, we compute the level density for the real and the complex correlated Jacobi model. We switch in section 4.4 to the real correlated Wishart model, consider analytically and numerically the k-point function and the level density, respectively, and derive the desired asymptotic relation. We close this chapter with a summery in section 4.5.

4.1 Smallest Eigenvalue Statistics

In the previous chapter, we discussed the statistics of the smallest eigenvalue within the uncorrelated Wishart model. We studied in detail the real and the real quaternion case. In this section, we present an analysis of its statistics for the more involved real and complex model possessing a non-trivial correlation structure. We derive for finite values of n, p and in the microscopic limit an analytic closed-form expression for the distribution of the smallest eigenvalue and the related gap probability in terms of a Pfaffian or a determinantal structure and known functions. The analysis of the smallest eigenvalue statistics presented here is based on Refs. [1,2] and extends some of the aspects discussed therein.

Other approaches to study the smallest eigenvalue statistics are available in the literature. For a general β the authors of Ref. [141,142] derived an expression for the gap probability in terms of a finite series of Jack polynomials. It has the drawback that further analytical considerations are hardly possible, the polynomials have to be constructed on a computer [103] and it does neither yield new insights into its dependence on the empirical eigenvalues nor does it show any structure like a Pfaffian or a determinant. Moreover, in Ref. [143] and Refs. [144, 145] the authors computed independently in the complex ensemble the gap probability and the distribution of the smallest eigenvalue, respectively, exploiting the given closed-form expression for the unitary Itzykson-Zuber integral. What is not consider are the statistics of the smallest eigenvalue with other kinds of correlations as in Refs. [130, 146, 147].

We derive mutual dualities between observables within the correlated Wishart model in section 4.1.1 and show that the gap probability to find all eigenvalues of a Wishart matrix above a threshold s can be written as an averaged characteristic polynomial to some power. In section 4.1.2 we extend this duality and construct invariant ordinary and supermatrix models dual to the gap probability. Employing these models, we derive in section 4.1.3 a closed-form expression for the smallest eigenvalue distribution as well as the gap probability. In section 4.1.4, we perform an asymptotic analysis of the gap probabilities and show that the smallest eigenvalue statistics are universal and independent of the empirical eigenvalues. We close this section with a comparison between our analytic results and Monte-Carlo simulations in section 4.1.5.

4.1.1 A Duality of Ordinary Wishart Models

It is worth mentioning that the smallest eigenvalue distribution in the case of the correlated Wishart ensemble can be studied by analyzing the related gap probability (2.43) only.

In section 2.3.3, we construct a full matrix model average for the gap probability to find all eigenvalues above a threshold s within the real and complex correlated Wishart model. It was obtained using the Θ-function and is given in Eq. (2.44). From this representation it is not obvious that supersymmetry or bosonization is suitable to study this quantity. Thus, we start our discussion by considering Eq. (2.43). To eliminate a Θ-function, we shift all eigenvalues by s and find

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = \exp\left(-\frac{\beta s}{2}\mathrm{tr}\Lambda^{-1}\right)\int \mathrm{d}[X]\ \ |\Delta_p(X)|^\beta \\ \times \det{}^{\beta(n-p+1-2/\beta)/2}(X+s\mathbf{1}_p)\Phi_\beta(X,\Lambda^{-1})\ . \end{aligned} \tag{4.1}$$

Because no closed-form expression in terms of known functions exists for the orthogonal Itzykson-Zuber integral Φ_1, we can not analyze the gap probability (4.1) further. Since in Eq. (4.1) we are averaging over a characteristic polynomial to some power, supersymmetry is the method of choice. To apply it, we have to find an underlying dual Wishart matrix model such that diagonalizing it yields Eq. (4.1). A naive choice is the original model consisting of $p\times n$ dimensional data matrices W.

To construct such a matrix ensemble, it is reasonable to study the decomposition of the volume element for a general rectangular matrix. We take $\widehat{W}$ to be a complex or a real $M\times N$ matrix ($N\geq M$) with the flat measure $\mathrm{d}[\widehat{W}]$, given by the product of all independent differentials. Diagonalization $\widehat{W}\widehat{W}^\dagger = UXU^\dagger$, where $X=\mathrm{diag}(x_1,\ldots,x_M)$ are the positive eigenvalues and $U\in \mathrm{O}(M),\mathrm{U}(M)$, induces a decomposition of the volume form given by [41, 44]

$$\mathrm{d}[\widehat{W}]\ \sim |\Delta_M(X)|^\beta \det{}^{\beta(N-M+1-2/\beta)/2}X\,\mathrm{d}[X]\ \mathrm{d}\mu(U)\ , \tag{4.2}$$

where $\mathrm{d}\mu(U)$ is the Haar measure. Hence, the main obstacle when mapping Eq. (4.1) to a model with $M=p$ and $N=n$ is that the shift by s destroyed parts of the Jacobian (4.2). We introduce the missing monomial factor in Eq. (4.1) by inserting an one

$$1=\frac{\prod_{i=1}^p x_i^{\beta(n-p+1-2/\beta)/2}}{\prod_{i=1}^p x_i^{\beta(n-p+1-2/\beta)/2}}=\frac{\prod_{i=1}^p x_i^{\beta(n-p+1-2/\beta)/2}}{\det^{\beta(n-p+1-2/\beta)/2}X}\ . \tag{4.3}$$

This completes the Jacobian in the gap probability (4.1) and we are left with an averaged ratio of characteristic polynomials representing the gap probability,

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K\exp\left(-\mathrm{tr}\frac{\beta s}{2\Lambda}\right)\int \mathrm{d}[W]\ \exp\left(-\frac{\beta}{2}\mathrm{tr}WW^\dagger\Lambda^{-1}\right) \\ \times\frac{\det^{\beta(n-p+1-2/\beta)/2}(WW^\dagger+s\mathbf{1}_p)}{\det^{\beta(n-p+1-2/\beta)/2}WW^\dagger}\ , \end{aligned} \tag{4.4}$$

where we use W instead of $\widehat{W}$, as we are back in the original Wishart model. In expression (4.4) we denote by K an overall normalization constant determined by the condition that $E_p^{(\beta)}([0,s];0)\to 1$ for $s\to 0$. If we compare Eq. (4.4) with Eq. (2.99), we can read off a supermatrix model dual to the gap probability (4.4). For the complex correlated Wishart ensemble we find

$$\begin{aligned} E_p^{(2)}([0,s];0) = K\exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right)\mathrm{sdet}^{n-p}\kappa\int \mathrm{d}[\sigma]\ \exp\left(\imath\mathrm{str}\kappa\sigma\right) \\ \times\,\mathrm{sdet}^{-1}\left(\mathbf{1}_p\otimes\mathbf{1}_{\nu|\nu}-\imath\Lambda\otimes\sigma\right)I_2\left(\sigma\right)\ , \end{aligned} \tag{4.5}$$

where $\kappa = \text{diag}(\varepsilon \mathbf{1}_\nu, s\mathbf{1}_\nu)$ and $\nu = n - p$. At the end of the calculation we take the limit $\varepsilon \to 0$. The integration domain is the space of Hermitian supermatrices of dimension $(\nu|\nu) \times (\nu|\nu)$. For the average (4.4) over real matrices W, we distinguish between ν even and odd, because for the former the determinants in Eq. (4.4) have a half-integer power. We begin with ν odd and find

$$\begin{aligned} E_p^{(1)}([0,s];0) = K \exp\left(-\text{tr}\frac{s}{2\Lambda}\right) \text{sdet}^{(n-p)/2}\kappa \int \mathrm{d}[\sigma] \ \exp\left(\frac{\imath}{2}\text{str}\kappa\sigma\right) \\ \times \,\text{sdet}^{-1/2}\left(\mathbf{1}_p \otimes \mathbf{1}_{(\nu-1)|(\nu-1)} - \imath\Lambda \otimes \sigma\right) I_n(\sigma) \ , \end{aligned} \tag{4.6}$$

where $\kappa = \text{diag}(\varepsilon\mathbf{1}_{(\nu-1)}, s\mathbf{1}_{(\nu-1)})$. The integration domain is the space of Hermitian supermatrices with a symmetric matrix in the boson-boson and a self-dual matrix in the fermion-fermion block, see section 2.5. If $\nu = n - p$ is even, the characteristic polynomial has half-integer power. In this case, we complete the determinant in the numerator to an integer power

$$\text{det}^{(\nu-1)/2}(WW^\dagger + s\mathbf{1}_p) = \frac{\text{det}^{\nu-1}(WW^\dagger + s\mathbf{1}_p)}{\text{det}^{(\nu-1)/2}(WW^\dagger + s\mathbf{1}_p)} \tag{4.7}$$

and apply the results of section 2.5. This leads to the same formula as for odd ν, but with $\kappa = \text{diag}(\varepsilon\mathbf{1}_{\nu-1}, s\mathbf{1}_{\nu-1}, s\mathbf{1}_{2(\nu-1)})$ and σ of dimension $(2(\nu-1)|2(\nu-1)) \times (2(\nu-1)|2(\nu-1))$. Instead of doubling the determinants in the numerator of Eq. (4.7), we could as well move $\text{det}^{-1/2}(WW^\dagger + s\mathbf{1}_p)$ from the numerator into the denominator. This leads to a supermatrix model in the same symmetry class as Eq. (4.6) of size $(\nu|\nu) \times (\nu|\nu)$. It isn't obvious, but both models are related by integral theorems on supermanifolds [64, 148–152].

Since for $\beta = 2$ the supersymmetric Itzykson-Zuber integral and the Efetov-Wegner terms are known, see Refs. [153, 154], the supermatrix model corresponding to the gap probability in the complex correlated Wishart ensemble (4.5) can be computed exactly. Unfortunately, this is not the case for the much more relevant case of the real ensemble.

To this end, we establish an approach circumventing this difficulty for both ensembles in a unified way. By analyzing the requirements a model dual to Eq. (4.1) has to satisfy, we obtain that this does not fix N. It only fixes $M = p$ and bounds $N \geq M$. Thus, we obtain that

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K \exp\left(-\text{tr}\frac{\beta s}{2\Lambda}\right) \int \mathrm{d}[\widehat{W}] \ \exp\left(-\frac{\beta}{2}\text{tr}\widehat{W}\widehat{W}^\dagger\Lambda^{-1}\right) \\ \times \frac{\text{det}^{\beta(n-p+1-2/\beta)/2}(\widehat{W}\widehat{W}^\dagger + s\mathbf{1}_p)}{\text{det}^{\beta(m+1-2/\beta)/2}\,\widehat{W}\widehat{W}^\dagger} \ , \end{aligned} \tag{4.8}$$

where $\widehat{W}$ is a $p \times (p + m)$ dimensional matrix with $\widehat{W}_{ij} \in \mathbb{R}, \mathbb{C}$ and $m \in \mathbb{N}_0$. Since we obtain infinitely many possibilities, one could ask "*does their exist a best choice for m?*". The answer is: Yes. Indeed, if we take $m = 2/\beta - 1$, the determinant in the denominator disappears and we are left with an averaged characteristic polynomial

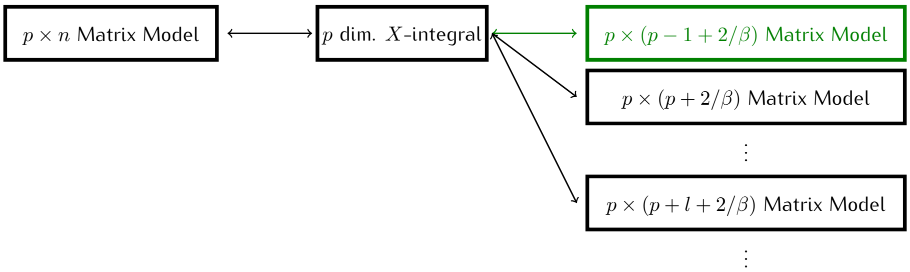


Figure 4.1: The different dual matrix models obtained so far. The green highlighted model is the "best choice".

to some power

$$E_p^{(\beta)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{\beta s}{2\Lambda}\right) \int d[\widehat{W}]\ \exp\left(-\frac{\beta}{2}\mathrm{tr}\widehat{W}\widehat{W}^\dagger\Lambda^{-1}\right) \\ \times \det{}^{\beta(n-p+1-2/\beta)/2}(\widehat{W}\widehat{W}^\dagger + s\mathbf{1}_p)\ , \tag{4.9}$$

where $\widehat{W}$ is fixed to be a $p \times (p-\beta+2)$ matrix. The advantage of this particular choice becomes clear if one compares the expression above with the results of section 2.5. It will lead to an invariant ordinary Hermitian matrix model. In Fig. 4.1, we summarize the dualities we found so far .

The duality (4.8) is part of a large class of dualities, which cover all kinds of partially invariant distribution functions and observables. We take B and $\widehat{B}$ to be a $M \times N$, respectively, a $M \times (N-l)$ dimensional matrix, such that $N-l \geq M$ with $l \in \mathbb{N}$ arbitrary. We consider two integrable functions $f_1(BB^\dagger\Lambda^{-1})$ and $f_2(BB^\dagger)$ which are smooth and invariant such that the integral in Eq. (4.10) exists. Invariant means that f does not change under the transformation $f_i(A) \mapsto f_i(UAU^\dagger)$ with either $U \in \mathrm{U}(M)$ if $\beta = 2$ or $U \in \mathrm{O}(M)$ if $\beta = 1$. We find

$$\int d[B]\ \frac{f_2\left(BB^\dagger\right)}{\det^{l\beta/2}\widehat{B}\widehat{B}^\dagger}\ f_1(BB^\dagger\Lambda^{-1}) \\ = \frac{\mathrm{Vol}\left(\mathrm{U}(N)\right)}{\mathrm{Vol}\left(\mathrm{U}(N-l)\right)} \int d[\widehat{B}]\ f_2\left(\widehat{B}\widehat{B}^\dagger\right)\ f_1\left(\widehat{B}\widehat{B}^\dagger\Lambda^{-1}\right)\ , \tag{4.10}$$

where the Vol(U(N))/Vol (U($N-l$)) is independent of the normalization of the Haar measure.

In the literature other dualities are known, for instance, in Chiral Random Matrix Theory a duality known as "flavor-topology duality" was found [56,90,155]. It states that the topological charge $\nu = |N-M|$ can be interpreted as ν additional massless flavor degrees of freedom. For $\beta = 2$ let $\widehat{B}$ be a complex $M \times N$ matrix, then the flavor-topology duality means that

$$\begin{aligned} Z^{\beta=2}_{N_f,\nu}(m_1,\ldots,m_f) &= \int \mathrm{d}[\widehat{B}] \prod_{f=1}^{N_f} \det\left(\mathcal{D}+m_f^2\mathbf{1}_{M+N}\right)\exp\left(-\frac{M}{2}\mathrm{tr}\widehat{B}\widehat{B}^\dagger\right) \\ &\sim \int \mathrm{d}[\widehat{B}] \det{}^{\nu}\mathcal{D}\prod_{f=1}^{N_f} \det\left(\mathcal{D}+m_f^2\mathbf{1}_{2M}\right)\exp\left(-\frac{M}{2}\mathrm{tr}\widehat{B}\widehat{B}^\dagger\right) \\ &= Z^{\beta=2}_{N_f+\nu,0}(m_1,\ldots,m_f,0,\ldots,0)\ , \end{aligned} \tag{4.11}$$

where N_f is the number of different flavors, m_f are the flavor masses and

$$\mathcal{D} = \begin{bmatrix} 0 & \imath B \\ \imath B^\dagger & 0 \end{bmatrix} . \tag{4.12}$$

From the first to the second line in Eq. (4.11), we reduced the dimension of the matrix $\widehat{B}$ at the cost of an additional determinant to a power of ν coming from the Jacobian. The relation between the two dualities becomes clear if one compares Eq. (4.11) and Eq. (4.10). At the beginning we have a matrix model of dimension $M \times N$ which we reduced to a matrix model of dimension $p \times (M-N-l)$.

The difference between both dualities is that we start with a determinant in the denominator and therefore decrease the dimensionality. Another crucial difference in our case is the presence of correlations. Thus, our and the flavor-topology duality are based on the same freedom in the Wishart model only.

4.1.2 Dual Invariant Models in Ordinary and Superspace

Based on the previous section and results of section 2.5, we present two invariant matrix models dual to Eq. (4.9). We obtain two different models, because we have to distinguish for the real ensemble between even and odd values of $n-p$. For even $n-p$, we obtain an invariant supermatrix model whereas in all other cases, we obtain an invariant ordinary matrix model. We introduce

$$v = \beta(n-p+1-2/\beta)/2 \tag{4.13}$$

to denote the exponent of the characteristic polynomial in the second line of Eq. (4.9) and distinguish for $\beta=1$ between integer and half-integer v.

A Dual Ordinary Matrix Model for Integer v

The condition of integer v is equivalent to studying Eq. (4.9) for $n-p$ odd if $\beta=1$ and all n,p if $\beta=2$. In this case the power of the characteristic polynomial is integer and we do not need full supersymmetry to analyze the gap probability. Instead of a supervector, we use only complex vectors with Grassmannian entries in Eq. (2.101). Following the reasoning in section 2.5, we arrive at

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{\beta s}{2\Lambda}\right) \int \mathrm{d}[\sigma]\ \exp\left(\mathrm{tr}\sigma\right) f_{p+2-\beta,\beta}(\sigma) \\ \times \prod_{k=0}^{p} \mathrm{det}^{\beta/2}\left(\frac{\beta s}{2}\mathbf{1}_{2v/\beta} + \Lambda_k\sigma\right) \ , \end{aligned} \tag{4.14}$$

where σ is a Hermitian matrix of dimension $2v/\beta \times 2v/\beta$. If $\beta = 1$ it possesses a further symmetry, it is self-dual. The function $f_{m,\beta}$ is invariant, *i.e.* $f_{m,\beta}(u\sigma u^\dagger) = f_{m,\beta}(\sigma)$ for $u \in \mathrm{USp}(2v), \mathrm{U}(v)$ for $\beta = 1, 2$, respectively. For $m \in \mathbb{N}$ it is a distribution on the space of Hermitian matrices and it is given as the Fourier transform of a characteristic polynomial

$$f_{m,\beta}(\sigma) = \int \mathrm{d}[\varrho]\ \mathrm{det}^{\beta m/2}\varrho\ \exp\left(-\imath\mathrm{tr}\varrho\sigma\right) \ , \tag{4.15}$$

where ρ is in the same symmetry class as σ. Thus, since $f_{m,\beta}$ as well as the remaining integrand are invariant under change of basis, we arrive at an invariant matrix model such that we can diagonalize σ and study an eigenvalue integral. Due to the degeneracy in the eigenvalues of the self-dual matrix for $\beta = 1$, all square roots in Eq. (4.14) disappear such that we can solve the eigenvalue integral for $\beta = 1, 2$ using random matrix theory techniques.

Instead of the generalized Hubbard-Stratonovich transformation, we can use ordinary bosonization [156] and find

$$E_p^{(\beta)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{\beta s}{2\Lambda}\right) t^{\gamma(\beta-2)} \int \mathrm{d}\mu(U)\mathrm{det}^{-(p+2-\beta)/2}U \tag{4.16}$$

$$\times \exp\left(-\imath\frac{\beta s}{2}\mathrm{tr}U\right) \prod_{k=1}^{p} \mathrm{det}^{-\beta/2}\left(\mathbf{1}_{2\gamma/\beta} - \imath\Lambda_k U\right) \ . \tag{4.17}$$

The matrix U is either in the circular symplectic $\mathrm{U}(2v)/\mathrm{USp}(2v)$ or the circular unitary ensemble $\mathrm{U}(v)$ [40]. The measure $\mathrm{d}\mu(U)$ is induced from the Haar measure on the space of unitary matrices and is related to the flat measure by

$$\mathrm{d}\mu(U) \sim \mathrm{d}[U]\ \mathrm{det}^{-v-\frac{\beta-2}{2}}U \ . \tag{4.18}$$

Since U possesses a degeneracy in the eigenvalue spectrum, all square roots disappear. We compute the resulting expression analytically by going into its eigenvalue basis. It is a matter of taste which method is preferred. As we will see later, Eq. (4.17) is used to analyze the microscopic limit, because it is easier to study its asymptotic behavior.

A Dual Supermatrix Model for Half Integer v

The power of the characteristic polynomial in Eq. (4.9) can be half-integer, but only for $\beta = 1$ and $\nu = n-p$ even. We can not simply use a Gaussian integral over vectors with real Grassmannian entries, but we can circumvent this difficulty by completing the expression to integer power at the cost of an additional determinant in the denominator, *i.e.*

$$\det{}^{\alpha+1/2}\left(WW^\dagger + s\mathbf{1}_p\right) \;\; = \frac{\det{}^{\alpha+1}\left(WW^\dagger + s\mathbf{1}_p\right)}{\det{}^{1/2}\left(WW^\dagger + s\mathbf{1}_p\right)} \;, \tag{4.19}$$

where we introduced $2\alpha = \nu - 2$. The determinant in the numerator is handled using Gaussian integrals over complex vectors with Grassmannian entries and the determinant in the denominator with Gaussian integrals over a real vector. Thus, employing the results of section 2.5 we arrive at an unusual supermatrix model

$$\begin{aligned} E_p^{(1)}(s) = K \exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right) \int \mathrm{d}[\mu] \;\; \exp\left(\imath \mathrm{str}\mu\right) \;\; I_p(\mu) \\ \times \prod_{k=1}^{p} \;\mathrm{sdet}^{-1/2}\left(\frac{s}{2}\mathbf{1}_{1|\nu} - \imath\Lambda_k\mu\right) \;. \end{aligned} \tag{4.20}$$

The matrix μ is a supermatrix with a fermion-fermion block of size $\nu \times \nu$ given by a real quaternion self-dual matrix σ and a one dimensional bosonic-bosonic block y,

$$\mu = \left[\begin{array}{ccc} y & \eta^\dagger & \eta^T \\ \eta & & \\ -\eta^* & \multicolumn{2}{c}{\imath\sigma} \end{array} \right] \;. \tag{4.21}$$

In this parametrization η is a $\alpha + 1$ dimensional complex vector with Grassmannian entries. The measure is the flat one, *i.e.* the product of all independent differentials, and $I_p(\mu)$ is the supersymmetric Ingham-Siegel integral, see Eq. (2.97). What makes this supermatrix model unusual is the fact that although it is invariant, it can not be reduced by any integral theorem.

Since the Efetov-Wegner terms for real supermatrix models are unknown, the only possibility to solve this integral is to expand it in the Grassmann variables, do the Berezinian integrals first such that diagonalization does not pose a problem. Since we don't know at present how all integrals can be done even in the simplest cases, we leave the exact calculation for a future project.

Synopsis

In the previous section, we obtained altogether four dual matrix models. Two ordinary ones of Wishart type, namely Eq. (4.4) and Eq. (4.9) which we call the large-W, respectively, the small-W model. Moreover, we found a dual invariant ordinary (super) matrix model, the small-σ model and a non-invariant supermatrix model, the large-σ model given by Eq. (4.14) (Eq. (4.20)), respectively, Eq. (4.6) (Eq. (4.5)). We summarize this four-fold duality schematically

$$
\begin{array}{cccc}
 & \nearrow \begin{array}{c}\textbf{large-}W\textbf{ model}\\ p\times n\text{-dim.}\end{array} & \Leftrightarrow & \begin{array}{c}\textbf{large-}\sigma\textbf{ model}\\ (2v/\beta|2v/\beta)\text{-dim.}\end{array}\\
E_p^{(\beta)}([0,s];0) & & & \\
 & \searrow \begin{array}{c}\textbf{small-}W\textbf{ model}\\ p\times \bar{n}\text{-dim.}\end{array} & \Leftrightarrow & \begin{array}{c}\textbf{small-}\sigma\textbf{ model}\\ 2v/\beta\times 2v/\beta\text{-dim.}\end{array}
\end{array}
$$

if $v \in \mathbb{N}$ and

$$
\begin{array}{cccc}
 & \nearrow \begin{array}{c}\textbf{large-}W\textbf{ model}\\ p\times n\text{-dim.}\end{array} & \Leftrightarrow & \begin{array}{c}\textbf{large-}\sigma\textbf{ model}\\ (\nu|\nu)\text{-dim.}\end{array}\\
E_p^{(1)}([0,s];0) & & & \\
 & \searrow \begin{array}{c}\textbf{small-}W\textbf{ model}\\ p\times \bar{n}\text{-dim.}\end{array} & \Leftrightarrow & \begin{array}{c}\textbf{small-}\sigma\textbf{ model}\\ (1|\nu)\text{-dim.}\end{array}
\end{array}
$$

if $\beta = 1$ and $v \in \frac{1}{2}\mathbb{N}$, where $\bar{n} = p+2-\beta$. It should be emphasized that this scheme is true for all kinds of invariant probability distributions. This is a consequence of the arguments leading to the correspondence in Eq. (4.10) and of section 2.5.

4.1.3 Exact Gap Probability and Smallest Eigenvalue Distribution

In this section, we derive a closed-form expression for the gap probability for all values of n, p if $\beta = 2$ and all values n, p with $n - p$ even if $\beta = 1$. The calculations are done for the complex and the real case separately. Because it is more intuitive, we start with the complex case and adjust subsequently the calculations to the real case. We complete this section by a derivation of the smallest eigenvalue distribution from the exact results for the gap probability in both cases.

Exact Expressions for the Complex Case

We obtained in the previous section an invariant Hermitian matrix model such that can diagonalize the integration variable $\sigma = u\hat{s}u^\dagger$ where $\hat{s} =$ diag$(\hat{s}_1, \ldots, \hat{s}_v)$, $u \in$ U(v) and $v = n - p$ is exactly the rectangularity of the data matrices. Applying the diagonalization to Eq. (4.14), we find

$$
\begin{aligned}
E_p^{(2)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right) \int \mathrm{d}[\hat{s}]\ \ |\Delta_v(\hat{s})|^2 \exp(\mathrm{tr}\hat{s}) \\
\times f_{p,2}(\hat{s}) \prod_{k=1,i=1}^{p,v} (s + \Lambda_k \hat{s}_i)\ .
\end{aligned}
\tag{4.22}
$$

To solve this eigenvalue integral, we need to know a closed-form expression for $f_{p,2}$. We repeat in appendix B.1 a similar calculation of Ref. [109]. There we show that $f_{p,2}$ is given by

$$
f_{p,2}(\hat{s}) \sim \prod_{i=1}^{v} \frac{\partial^{p+v-1}}{\partial \hat{s}_i^{p+v-1}} \delta(\hat{s}_i)\ . \tag{4.23}
$$

Because it factorizes in the eigenvalues $\hat{s}_i$, the calculation of the resulting eigenvalue integral when inserting Eq. (4.23) into Eq. (4.22) can be done using results from random matrix theory [40,44]. In appendix B.1, we show that if we apply these results the gap probability to find no eigenvalue of a complex correlated Wishart matrix within an interval $[0,s]$ is given by

$$E_p^{(2)}([0,s];0) = \frac{(-1)^{v(v-1)/2}}{\det^v \Lambda} \exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right) \det\left[\Theta(\alpha_2) \sum_{k=0}^{\min(p,\alpha_2)} \frac{s^{p-k} e_k(\Lambda)}{(\alpha_2 - k)!}\right] \ , \tag{4.24}$$

where $\alpha_2 = p + v + 1 - i - j$, $1 \leq i,j \leq v$ and $e_k(\Lambda)$ denotes the kth elementary symmetric function,

$$e_k(\Lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq p} \Lambda_{i_1} \cdots \Lambda_{i_k} \ . \tag{4.25}$$

Contrary to known results [141,143–145] the gap probability (4.24) has a determinantal structure of dimension $v = n - p$. This property makes the evaluation of the resulting expression fast as long as v is not too large. Moreover, every entry of the matrix kernel depends on the entire empirical eigenvalue spectrum.

Exact Expressions for the Real Case

For the gap probability to find no eigenvalue of a correlated real Wishart matrix within an interval $[0,s]$, we obtain an invariant model of real quaternion self-dual matrices, see Eq. (4.14). It is invariant under the action of USp$(2v)$. Because of the invariance, we can diagonalize the integration variable $\sigma = u(\mathbf{1}_2 \otimes \hat{s})u^\dagger$, where $u \in$ USp$(2v)$ and $\hat{s} = \mathrm{diag}(\hat{s}_1, \ldots, \hat{s}_v)$ is the matrix of distinct eigenvalues, such that

$$\begin{aligned} E_p^{(1)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right) \int \mathrm{d}[s]\ \Delta_v^4(\hat{s}) \exp\left(2\mathrm{tr}\hat{s}\right) \\ \times f_{p+1,1}(\mathbf{1}_2 \otimes \hat{s}) \prod_{k=0}^{p} \det\left(\frac{s}{2}\mathbf{1}_v + \Lambda_k \hat{s}\right) \ . \end{aligned} \tag{4.26}$$

The distinguishing feature of this eigenvalue integral is that we can not compute the $f_{p+1,1}$, because we do not know the unitary-symplectic Itzykson-Zuber integral. Accordingly, we have to use the results of Ref. [113], where the authors showed that

$$f_{p+1,1}(\mathbf{1}_2 \otimes \hat{s}) \sim \prod_{i=1}^{v} \frac{\partial^{p+2v-1}}{\partial \hat{s}_i^{p+2v-1}} \delta(\hat{s}_i) \ . \tag{4.27}$$

Similar to the complex case, the factorization of $f_{p+1,1}$ facilitates the computation of the resulting eigenvalue integral when substituting Eq. (4.27) intro Eq. (4.26). In appendix B.2, we show that the gap probability to find no eigenvalue of a real correlated Wishart matrix has a Pfaffian structure and is given by

$$E_p^{(1)}([0,s];0) = \frac{\exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right)}{\det^v \Lambda} \mathrm{pf}\left[\sum_{k=0}^{\min(p,\alpha_1)} \frac{(j-i)\Theta(\alpha_1)\ e_k(\Lambda)}{(\alpha_1 - k)!} s^{p-k}\right] \ , \tag{4.28}$$

where $\alpha_1 = p+2v-i-j+2$ and $1 \leq i,j \leq 2v$. The gap probability (4.28) shows for the first time that even for the real correlated Wishart model, an integrable Pfaffian structure exists. It is evidence that, at least in some cases it is possible to fully circumvent the orthogonal Itzykson-Zuber integral occurring in the joint probability distribution function (2.20). Moreover, it turns out, that apart from an exponential prefactor, the gap probability is even for the real ensemble polynomial in s and Λ.

Distribution of the Smallest Eigenvalue

It was shown in the introduction that if we know the gap probability, we know the distribution of the smallest eigenvalue, which is given as the derivative of the gap probability with respect to the threshold parameter s, see Eq. (2.42),

$$\mathcal{P}^{(\beta)}_{\mathrm{min}}(s) = -\frac{\mathrm{d}}{\mathrm{d}s} E^{(\beta)}_p([0,s];0) \ . \tag{4.29}$$

For the smallest eigenvalue distribution of the complex Wishart matrix, we obtain

$$\begin{aligned}\mathcal{P}^{(2)}_{\mathrm{min}}(s) &= \mathrm{tr}\Lambda^{-1} E^{(2)}_p([0,s];0) \\ &- \frac{(-1)^{v(v-1)/2}}{\det^v \Lambda} \exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right) \sum_{i=1}^{v} \det\left[\Theta(\alpha_2)\mathcal{G}^{(l)}_{ij}(s)\right]\end{aligned} \tag{4.30}$$

where

$$\mathcal{G}^{(l)}_{ij}(s) = \begin{cases} \displaystyle\sum_{k=0}^{\min(p-1,\alpha_2)} \frac{(p-k)s^{p-k-1}e_k(\Lambda)}{(\alpha_2-k)!} & ,l=i \\ \displaystyle\sum_{k=0}^{\min(p,\alpha_2)} \frac{s^{p-k}e_k(\Lambda)}{(\alpha_2-k)!} & ,l \neq i \end{cases} , \tag{4.31}$$

and $1 \leq i,j \leq v$. For the real correlated Wishart ensemble, the smallest eigenvalue distribution can readily be derived from the gap probability (4.28) and is given by

$$\mathcal{P}^{(1)}_{\mathrm{min}}(s) = \mathrm{tr}\frac{1}{2\Lambda} E^{(1)}_p([0,s];0) - \frac{\exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right)}{\det^v \Lambda} \sum_{i=1}^{2v} \mathrm{pf}\left[\Theta(\alpha_1)\mathcal{G}^{(l)}_{ij}(s)\right] \tag{4.32}$$

where

$$\mathcal{G}^{(l)}_{ij}(s) = \begin{cases} \displaystyle\sum_{k=0}^{\min(p-1,\alpha_1)} \frac{(p-k)s^{p-k-1}e_k(\Lambda)}{(\alpha_1-k)!} & ,l=i \\ \displaystyle\sum_{k=0}^{\min(p,\alpha_1)} \frac{s^{p-k}e_k(\Lambda)}{(\alpha_1-k)!} & ,l \neq i \end{cases} , \tag{4.33}$$

and $1 \leq i,j \leq 2v$. In contrast to the uncorrelated Wishart model discussed in section 3.1 and 3.2 the derivative destroys the Pfaffian and determinantal structure such that we obtain a sum of Pfaffians and determinants.

4.1.4 Microscopic Limit

The ordinary matrix model (4.14) and the supermatrix model (4.20) for the gap probability, are eminently suitable for the microscopic limit, because in this limit n, p tend to infinity while $\nu = n - p$ is kept fixed. Thus, the dimension of the dual invariant model does not vary. Hence, we can analyze this limit by studying the asymptotic behavior of the integrands. For the uncorrelated Wishart model, the microscopic limit of the distribution of the smallest eigenvalue and the related gap probability are introduced as [91]

$$\wp_{\text{min}}^{(\beta)}(u) = \lim_{p\to\infty} \frac{1}{4p} \mathcal{P}_{\text{min}}^{(\beta)}\left(\frac{u}{4p}\right) , \tag{4.34}$$

respectively,

$$\mathcal{E}^{(\beta)}(u) = \lim_{p\to\infty} E_p^{(\beta)}\left(\frac{u}{4p}\right) . \tag{4.35}$$

To study a similar limit in the correlated case, we analyze the expectation value of the smallest eigenvalue and how it scales with p if n, p tend to infinity with $\nu = n - p$ fixed. We do this for the even v case only, because the half-integer v case works analogously by simply replacing all invariants in the ordinary space with invariants in superspace. These findings will help us to provide the asymptotic behavior of the dual matrix models for both integer and non-integer v. We compute the limiting smallest eigenvalue distribution and gap probability for integer v using the asymptotic formulas.

Analysis of the Microscopic Limit

To compute the limiting distribution and gap probability, we analyze the mean behavior of the smallest eigenvalue. This leads us to the local scale on which we study both quantities. The scaling is derived from the mean of the smallest eigenvalue if we use

$$\langle s\rangle = \int\limits_0^\infty \mathrm{d}s\ s\ \mathcal{P}_{\text{min}}^{(\beta)}(s) = \int\limits_0^\infty \mathrm{d}s\ E_p^{(\beta)}([0,s];0) . \tag{4.36}$$

We express the gap probability on the right hand side of Eq. (4.36) by the matrix model obtained from bosonization (4.17). It has the advantage that we can omit an asymptotic analysis of the invariant distribution $f_{m,\beta}$, which has to be done for the model in Eq. (4.14). Inserting Eq. (4.17) into Eq. (4.36) yields

$$\begin{aligned} \langle s\rangle = K \int\limits_0^\infty \mathrm{d}s\ \exp\left(-\mathrm{tr}\frac{\beta s}{2\Lambda}\right) \int \mathrm{d}\mu(U) \exp\left(\mathrm{tr}U\right) \\ \times \det{}^{-\beta(p+2-\beta)/2} U \prod_{k=1}^{p} \det{}^{\beta/2}\left(\frac{\beta s}{2}\mathbf{1}_{2v/\beta} + \Lambda_k U\right) , \end{aligned} \tag{4.37}$$

where $d\mu(U)$ is independent of p and $U \in \mathrm{CSE}(2v) = \mathrm{U}(2v)/\mathrm{USp}(2v)$ or $U \in \mathrm{CUE}(v) = \mathrm{U}(v)$. Compared to Eq. (4.17), we rescaled U in Eq. (4.37) by $\imath 2/(\beta s)$. Because the normalization constant is independent of p,

$$K = \int d\mu(U) \exp(\mathrm{tr} U) \det{}^{-(2-\beta)/2} U \ , \tag{4.38}$$

the p dependent parts of the integrand are the exponent depending on s and Λ in the first row and the determinants in the second row of Eq. (4.37). The smallest eigenvalue statistics depend in a non-trivial way on the empirical eigenvalues. Therefore, the scaling determined in the following will depend on the empirical eigenvalues. On these grounds, it is reasonable to put some conditions on the set of empirical eigenvalues. We assume that almost all empirical eigenvalues Λ_k are of order $\mathcal{O}(1)$, with a finite number of order $\mathcal{O}(p^m)$ and none of the order $\mathcal{O}(p^{-m'})$ for $m, m' > 0$.

In the mathematical literature, Refs. [157–160] and references therein, the empirical eigenvalues Λ_k are treated, in a kind of "mean field approximation", as random variables respecting the distribution $\rho_{\mathrm{emp}}(\Lambda)$. In terms of this distribution, the above requirement can be interpreted as

$$\frac{1}{p}\sum_{i=1}^{p} f\left(\frac{1}{\Lambda_i}\right) \to \int d\Lambda \rho_{\mathrm{emp}}(\Lambda) f\left(\frac{1}{\Lambda}\right) \ , \tag{4.39}$$

for p large enough and $f \sim \mathcal{O}(1)$. This is a modest assumption, because system specific features are commonly encoded in the large eigenvalues and not in eigenvalues tending to zero when increasing the system size. Employing this assumption, we find

$$\frac{p}{\Lambda_{\mathrm{max}}^m} \leq \mathrm{tr}\frac{1}{\Lambda^m} \leq \frac{p}{\Lambda_{\mathrm{min}}^m}, \tag{4.40}$$

for all $m \in \mathbb{N}$. For the following asymptotic analysis, we assume that the empirical eigenvalues satisfy the above requirement. To study the large p behavior of the p-fold product in Eq. (4.36), we write the determinant as an exponent of a trace of a logarithm and expand the latter, yielding

$$\prod_{k=1}^{p} \det{}^{\beta/2}\left(\mathbf{1}_p + \frac{\beta s}{2\Lambda_k} U^\dagger\right) = \exp\left(-\frac{\beta}{2}\sum_{m=1}^{\infty}\frac{1}{m}\mathrm{tr}\left(\frac{-\beta s}{2\Lambda}\right)^m \mathrm{tr}(U^\dagger)^m\right) \ . \tag{4.41}$$

From the estimate (4.40) it turns out that $s^m \mathrm{tr}\Lambda^{-m}$ is of order $\mathcal{O}(p)$ for all m if $s \sim \mathcal{O}(1)$. Hence, if we rescale the threshold parameter s by $1/(4p\eta)$, where the 4 is convention [91] and η is yet to be determined, $s^m \mathrm{tr}\Lambda^{-m}$ is of order $\mathcal{O}(p^{1-m})$. Introducing $u = 4p\eta s$, the analysis above shows that if we set

$$\eta = \frac{1}{p}\sum_{i=1}^{p}\frac{1}{\Lambda_i} \tag{4.42}$$

we obtain that in the microscopic limit the mean value

$$\langle u \rangle = K \int_0^\infty \mathrm{d}u \ \exp\left(-u\beta/8\right) \int \mathrm{d}\mu(U) \mathrm{det}^{-(2-\beta)/2} U \exp\left(\mathrm{tr}U + \frac{\beta^2 u}{16}\mathrm{tr}U^\dagger\right) \tag{4.43}$$

is of order $\mathcal{O}(1)$ and independent of the empirical eigenvalues. We find the same scaling if $n-p$ is even for the supermatrix model (4.20).

Limiting Gap Probability and Smallest Eigenvalue Distribution

We use the results just obtained to compute both the asymptotic behavior of the gap probability and therefore of the smallest eigenvalue distribution. We will see that on the scale $s = u/4p\eta$ these quantities are universal and identical for the correlated and the uncorrelated Wishart model. For the latter the microscopic limit of the distribution of the smallest eigenvalue as well as the gap probability are known, Refs. [3, 5, 91, 107, 130, 132, 133, 161] and section 3.1. Hence, to study the microscopic limit of the correlated model, we can utilize the universality and apply the results for the uncorrelated model.

Before doing so, we have to show that the statistics are universal and indeed agree. Employing the analysis of the previous paragraph, we derive the asymptotic behavior of the gap probability for integer and half-integer v. For v integer, we find that for $s \sim \mathcal{O}(p^{-1})$ it is given by

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K \exp\left(-\frac{\beta s p \eta}{2}\right) \int \mathrm{d}\mu(U) \mathrm{det}^{-\beta(2-\beta)/2} U \\ \times \exp\left(\mathrm{tr}U + \frac{\beta s \eta p}{2}\mathrm{tr}U^\dagger\right) + \mathcal{O}(p^{-1}) \ . \end{aligned} \tag{4.44}$$

and analogously for half-integer v. It turns out that the asymptotic expression (4.44) does no distinguish whether we consider a correlated Wishart model with a general Λ or an uncorrelated Wishart model with variance $1/\eta$. We summarize our findings in the following statement.

Statement: *Assume that n, p tend to infinity, while $n-p$ is fixed, the empirical eigenvalues are of the order $\mathcal{O}(1)$ with a finite number of order $\mathcal{O}(p^\tau)$, where $\tau > 0$. Under these modest assumptions the dual ordinary and supermatrix models for the gap probability $E_p^{(\beta)}([0,s];0)$ given by Eqs. (4.14) and (4.20) behave asymptotically like the matrix models*

$$\begin{aligned} E_p^{(\beta)}([0,s];0) \sim \exp\left(-\mathrm{tr}\frac{s\beta}{2\Lambda}\right) \int \mathrm{d}[\sigma] \quad \exp\left(\mathrm{tr}\sigma\right) \\ \times \mathrm{det}^{p\beta/2}\left(\frac{s\beta}{2}\mathbf{1}_{2v/\beta} + \frac{1}{\eta}\sigma\right) \ f_{p+2-\beta,\beta}(\sigma) \end{aligned} \tag{4.45}$$

for $v \in \mathbb{N}$ and for $v \in \mathbb{N}/2$ as

$$
\begin{aligned}
E_p^{(1)}([0,s];0) \sim \exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right) \int \mathrm{d}[\mu]\ \exp\left(\imath \mathrm{str}\mu\right) \\
\times\ \mathrm{sdet}^{-p/2}\left(\frac{s}{2}\mathbf{1}_{1|\nu} - \frac{\imath}{\eta}\mu\right)\ I_p(\mu)
\end{aligned}
\tag{4.46}
$$

where $\eta = tr\Lambda^{-1}/p$.

Thus, on the scale $s = u/4p\eta$, the gap probability and therefore the statistics of the smallest eigenvalue are universal in the microscopic limit. Moreover, it does not see any empirical correlation structure such that the statistics of the smallest eigenvalue for the uncorrelated and correlated Wishart model are identical.

Microscopic Limit Using the Dual Model

Since the matrix model (4.44) is invariant under the action of an appropriate symmetry group, we diagonalize the integration variable $U = v(\mathbf{1}_{2/\beta} \otimes r)v^\dagger$, where $r = \mathrm{diag}(r_1, \ldots, r_v)$ is the matrix of distinct eigenvalues and $v \in \mathrm{USp}(2v), \mathrm{U}(v)$. The eigenvalues are all on the unit circle, $|r_i| = 1$, such that if we rescale them by $4\sqrt{u}/\beta$ we arrive at

$$
\mathcal{E}^{(\beta)}(u) = K \exp\left(-\frac{\beta u}{8}\right) \oint \mathrm{d}[r]\ \Delta_v^{4/\beta}(r) \mathrm{det}^{-2v/\beta} r \exp\left(\frac{\sqrt{u}}{2}\left(\mathrm{tr}r + \mathrm{tr}r^{-1}\right)\right)\ . \tag{4.47}
$$

Because except the Vandermonde determinant the entire integrand factorizes in the eigenvalues $\hat{s}_i$ standard results of random matrix theory apply. We employ these results in appendix B.3 and show that the microscopic limit of the gap probability is given by

$$
\mathcal{E}^{(\beta)}(u) = \exp\left(-\frac{\beta u}{8}\right) \mathrm{det}^{\beta/2}\left[q_{ij}\left(\frac{\beta\sqrt{u}}{2}\right)^{i+j+m+1} \mathrm{I}_{i+j+m+1}\left(\sqrt{u}\right)\right]\ , \tag{4.48}
$$

where $1 \le i, j \le 2v/\beta$, $m = -4 + \beta - 2v/\beta$ and $q_{ij} = (j - i)$ for $\beta = 1$ and $q_{ij} = (-1)^{i-1}$ for $\beta = 2$. From the gap probability, the distribution of the smallest eigenvalue can be readily derived, yielding

$$
\wp^{(\beta)}(u) = \frac{\beta}{8}\mathcal{E}^{(\beta)}(u) - \exp\left(-\frac{\beta u}{8}\right) \sum_{l=1}^{2v/\beta} \mathrm{det}^{\beta/2}\left[q_{ij}\mathcal{L}_{ij}^{(l)}(u)\right]\ , \tag{4.49}
$$

where i, j, m and q_{ij} are as above and

$$
\mathcal{L}_{ij}^{(l)}(u) = \begin{cases} \left(\frac{\beta\sqrt{u}}{2}\right)^{i+j+m+1} \mathrm{I}_{i+j+m+1}\left(\sqrt{u}\right) & , l \neq i \\ \left(\frac{\beta\sqrt{u}}{2}\right)^{i+j+m} \mathrm{I}_{i+j+m}\left(\sqrt{u}\right) & , l = i \end{cases} \tag{4.50}
$$

Similar to the derivation of an exact expression for the smallest eigenvalue distribution, see section 4.1.3, we do not obtain a Pfaffian nor determinantal structure but a sum of them.

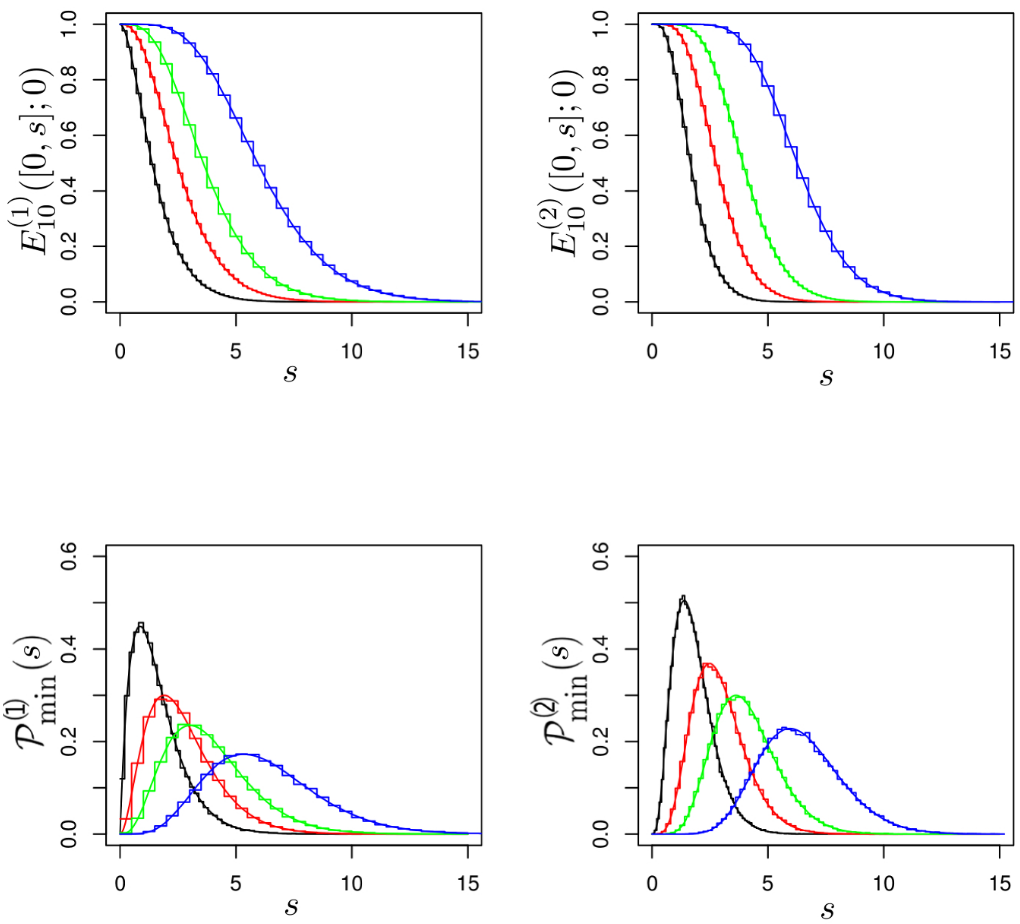


Figure 4.2: The upper two figures show $E_p^{(\beta)}([0,s];0)$ and the lower $\mathcal{P}_{\text{min}}^{(\beta)}(s)$ for fixed $p = 10$, where $\Lambda_k = 0.6, 1.2, 6.7, 9.3, 10.5, 15.5, 17.2, 20.25, 30.1, 35.4$ and $n = 13$ (black), $n = 15$ (red), $n = 17$ (green) and $n = 21$ (blue). The left figures correspond to the real ($\beta = 1$) and the right to the complex ensemble ($\beta = 2$). The lines correspond to our analytic results and the histograms to the numerical simulations.

4.1.5 Numerical Simulations

We compare our analytic findings, the exact as well as the limiting expressions, to numerical simulations for the purpose of illustration and to confirm the validity and correctness of our final expressions. We implement the exact formulas obtained for the gap probability (4.24) and (4.28) as well as the distribution of the smallest eigenvalue (4.30) and (4.32) into the computer code R [162] and generate $50\ 000$ complex and real correlated Wishart matrices drawn from the distribution (2.16). Since the rectangularity is the quantity which signifi-

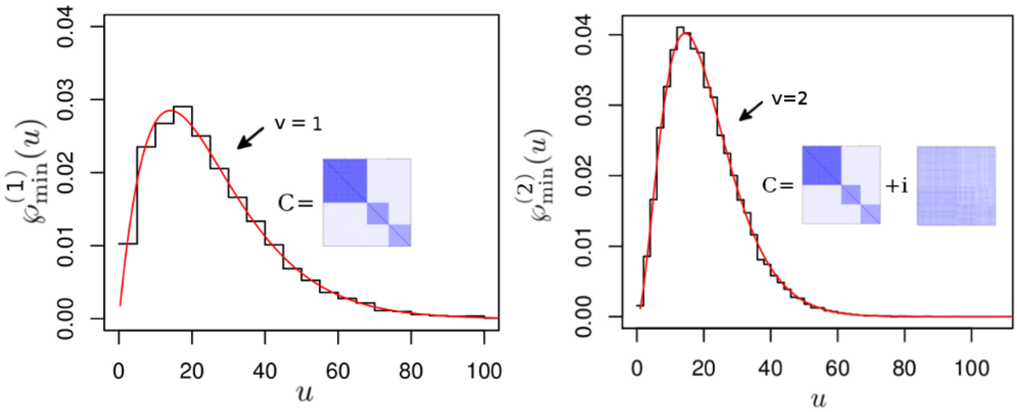


Figure 4.3: Left: The microscopic limit of the distribution of the smallest eigenvalue for $\beta = 1$ (line) compared to a numerical simulation (histogram) in a sample of real correlated 200×203 dimensional Wishart matrices with empirical correlation C indicated in the inset. Right: The microscopic limit of the distribution of the smallest eigenvalue for $\beta = 2$ (line) compared to a numerical simulation (histogram) in a sample of real correlated 200×201 dimensional complex Wishart matrices with empirical correlation C indicated in the inset.

cantly changes the shape and complexity of the analytic expressions, we carry out the simulations for four different rectangularities. The results are shown in Fig. 4.2. The empirical eigenvalues for both, the real and the complex samples are $\Lambda_k = 0.6, 1.2, 6.7, 9.3, 10.5, 15.5, 17.2, 20.25, 30.1, 35.4$. The figures show perfect agreement of the analytic formulas and the numerical simulations.

For the microscopic limit of the distribution of the smallest eigenvalue $\wp^{(\beta)}_{\text{min}}(u)$, we produce a non-trivial empirical correlation matrix and generate a sample of $30\ 000$ complex correlated 200×202-dimensional and real correlated 200×203-dimensional Wishart matrices. We indicate the non-trivial empirical correlation matrix C in the insets of Fig. 4.3. We obtain a perfect agreement of the numerical simulations and our analytic results shown in Fig. 4.3.

In section 4.1.4, we show that even for $\beta = 1$ and half-integer v, the gap probability and therefore the smallest eigenvalue distribution in the presence of correlations are universal. However, we did not derive analytics expressions. Because of the universality, we can analyze these statistics by means of the limit statistics in the uncorrelated model. The desired analytical expressions were derived in section 3.1 and are given by Eqs. (3.48) and (3.49). In Fig 4.4 we compare numerical simulations for a sample of $10\ 000$ 200×200 real Wishart correlation matrices, with $\nu = 0, 2, 4$ and an empirical correlation matrix indicated in the inset with our analytic results. We obtain a very good agreement for all values of ν. This not only confirms our findings concerning the universality of the real correlated Wishart model but also the correctness of the limiting smallest eigenvalue distribution for the uncorrelated real Wishart model with even rectangularity ν, computed in section 3.1.

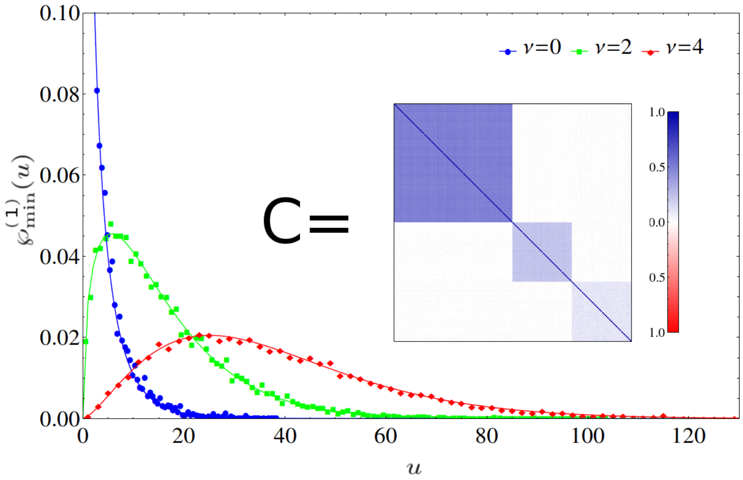


Figure 4.4: The analytical expressions for the microscopic limit of the smallest eigenvalue distribution for even $\nu = n - p$ in the real correlated Wishart ensemble (lines) compared to numerical simulations (dots).

4.2 Largest Eigenvalue Statistics

The cumulative density function of the largest eigenvalue within the real and complex correlated Wishart model is studied using the method of Grassmann variables and the Berezinian integral. It is the gap probability to find the largest eigenvalue of the model correlation matrix in an interval $[0, t]$ and therefore no eigenvalue above the threshold t. We exploit the positive definiteness of the Wishart correlation matrix to construct an invariant matrix model and analyze it further.

Except the final expression for the gap probability in the complex case computed in Ref. [143], none of the results neither the Wishart model for the gap probability nor the invariant matrix model derived in the upcoming sections are known in the literature. The only other attempt to study the gap probabilities for finite n, p and all values of β led to formulas in terms of hypergeometric functions of matrix argument [41, 141, 142]. These have the drawback to be evaluable on a computer only, see Ref. [103], and are hardly analyzable analytically.

In section 4.2.1, we construct an average over a full matrix model dual to the gap probability. This model is mapped in section 4.2.2 to an invariant matrix model using integral theorems on supermanifolds. In section 4.2.3, we derive an exact expression for the gap probability in the complex case and recover known results.

4.2.1 A Full Wishart Matrix Model Representation

We show that for the complex and the real correlated Wishart ensembles the largest eigenvalue statistics can be studied by applying the method of supersymmetry. This approach will lead to new insights which are not apparent from the analysis of section 5.2.

As explained in section 2.3.2, the largest eigenvalue distribution is related to the gap probability to find all eigenvalues of the Wishart matrix in an interval $[0,t]$ by a derivative of the latter with respect to t. Thus, it is the cumulative distribution function of the largest eigenvalue. Given the joint eigenvalue distribution function (2.20), it reads

$$E_p^{(\beta)}([0,t];p) = K\int\limits_0^t \mathrm{d}[X]\ P(X|\Lambda)\ , \tag{4.51}$$

where K is a normalization constant and $X = \mathrm{diag}(x_1,\ldots,x_p)$ is the matrix of eigenvalues of $WW^\dagger = UXU^\dagger$ with U in $\mathrm{O}(p)$, respectively, $\mathrm{U}(p)$ for $\beta = 1,2$. To apply supersymmetry we need a matrix model average over determinants, which means, a Wishart model such that diagonalization leads to Eq. (4.51). To find it, we need eigenvalue integrals over the positive real line $[0,\infty)$. Unfortunately, the integrals (4.51) are over a compact set $[0,t]$ only. Thus, simply shifting the eigenvalues as in the case of the smallest eigenvalue does not work. Therefore, we perform the coordinate change

$$x_i \to \frac{t}{x_i+1}\ \Leftrightarrow\ \mathrm{d}x_i \to \frac{-t\mathrm{d}x_i}{(x_i+1)^2}\ . \tag{4.52}$$

and obtain eigenvalue integrals over the entire positive real line

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = Kt^{np\beta/2}\int\limits_0^\infty \mathrm{d}[Y]\ |\Delta_p(Y)|^\beta \det{}^{-\tau\beta/2}\left(Y+\mathbf{1}_p\right) \\ \times\,\Phi_\beta((Y+\mathbf{1}_p)^{-1}|t\Lambda^{-1})\ , \end{aligned} \tag{4.53}$$

where Φ_β is the orthogonal or unitary Itzykson-Zuber integral and $\tau = n{+}p{+}2/\beta{-}1$. The drawback of the coordinate transformation is the inverse power of $Y+\mathbf{1}_p$ in the Itzykson-Zuber integral arising thereby.

Analogous to the smallest eigenvalue, we have infinitely many possibilities to construct a full matrix model average dual to Eq. (4.53), see section 4.1.1. By similar arguments, we choose

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = Kt^{np\beta/2}\int \mathrm{d}[\widehat{W}]\ \det{}^{-\tau\beta/2}\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p\right) \\ \times\exp\left(-\frac{\beta t}{2}\mathrm{tr}(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p)^{-1}\Lambda^{-1}\right)\ . \end{aligned} \tag{4.54}$$

as underlying Wishart model average, where $\widehat{W}$ is a $p\times\bar{n}$-dimensional real or complex data matrix and

$$\bar{n} = p-1+2/\beta = \left\{\begin{array}{ll} p+1 & ,\beta=1 \\ p & ,\beta=2 \end{array}\right.\ . \tag{4.55}$$

The analysis of section 2.5 implies that the average (4.54) has a dual invariant ordinary matrix model. However, the computation of the characteristic function (2.104) in the present case is highly non-trivial. We circumvent this difficulty by using

$$\lim_{N\to\infty}\frac{\det^N\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p-\frac{t\beta}{N2}\Lambda^{-1}\right)}{\det^N\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p\right)}=\exp\left(-\frac{\beta t}{2}\mathrm{tr}(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p)^{-1}\Lambda^{-1}\right) \tag{4.56}$$

and a projection formula developed in Ref. [139, 140], which will be explained in the next section. If we substitute expression (4.56) into Eq. (4.54) and exchange the N-limit with the $\widehat{W}$-integration, the resulting matrix model has a Lorentzian distribution,

$$E_p^{(\beta)}([0,t];p)=Kt^{np\beta/2}\lim_{N\to\infty}K_N\int d[\widehat{W}]\,\frac{\det^N\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p-\frac{t\beta}{2N}\Lambda^{-1}\right)}{\det^{N+\tau\beta/2}\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p\right)}\ . \tag{4.57}$$

The $\widehat{W}$-integral is convergent, because the exponent of the determinant in the denominator is always substantially larger than the exponent of the determinant in the numerator. The normalization constant K_N is introduced to absorb all N dependent prefactors. It is later determined by the condition that the leading order term in an asymptotic large N expansion should be of order $\mathcal{O}(1)$ such that the N-limit is finite.

4.2.2 Construction of a Dual Invariant Matrix Model

We construct an invariant matrix model dual to the average (4.57). We use the positive definiteness of $\widehat{W}\widehat{W}^\dagger$ and results of Refs. [139, 140]. The former manifests itself in the following identity

$$\det^{N+\tau\beta/2}\left(\widehat{W}\widehat{W}^\dagger+\mathbf{1}_p\right)=\det^{N+\tau\beta/2}\left(\widehat{W}^\dagger\widehat{W}+\mathbf{1}_{\bar{n}}\right)\ . \tag{4.58}$$

We express the determinant to a power N in the numerator of the integrand (4.57) as a Gaussian integral over N complex vectors with Grassmannian entries as introduced in Eq. (2.68). If we substitute this and the identity (4.58) into the gap probability (4.57), we arrive at

$$\begin{aligned}E_p^{(\beta)}([0,t];p)=K_{n,p}t^{np\beta/2}\lim_{N\to\infty}K_N\int d[\widehat{W},A]\,&\frac{\exp\left(\imath\mathrm{tr}\widehat{W}\widehat{W}^\dagger AA^\dagger\right)}{\det^{N+\tau\beta/2}\left(\widehat{W}^\dagger\widehat{W}+\mathbf{1}_{\bar{n}}\right)}\\ \times\exp\left(\imath\mathrm{tr}\left(\mathbf{1}_p-\frac{t\beta}{2N}\Lambda^{-1}\right)AA^\dagger\right)&\ ,\end{aligned} \tag{4.59}$$

where the structure of A depends on the symmetries of $\widehat{W}\widehat{W}^\dagger$. It is given by Eqs. (2.81) and (2.76) for $\beta=1,2$, respectively.

The integrand (4.59) is invariant under the action $\widehat{W}\mapsto\widehat{W}U$ with $U\in\mathrm{G}_{\bar{n}}$. Accordingly, we use the idea of Refs. [139,140] and extend the integration domain of the ordinary matrix $\widehat{W}$ to an integral over a rectangular supermatrix Σ by applying an integration theorem backwards. It states that the integral over the space of rectangular $(p+2N|2N)\times(\bar{n}|0)$ dimensional supermatrices of an invariant function $F(\Sigma^\dagger\Sigma)$ is given by

$$\int \mathrm{d}[\Sigma]\ F(\Sigma^\dagger\Sigma) \sim \int \mathrm{d}[\widehat{W}]\ F(\widehat{W}^\dagger\widehat{W}) = \int \mathrm{d}[\widehat{W}]\ f(\widehat{W}^\dagger\widehat{W})\ , \tag{4.60}$$

where f is a function on the space of ordinary matrices and the supermatrix Σ is given by

$$\Sigma = \left[\begin{array}{c} \widehat{W}_{ij} \\ \hline w_{ia}^* \\ \hline w_{ia} \\ \hline \chi_{ib} \end{array}\right] = \left[\begin{array}{c} \widehat{W}_{ij} \\ \hline w_{ia}^* \\ \hline w_{ia} \\ \hline -\xi_{ia}^* \\ \hline \xi_{ia} \end{array}\right]\ , \tag{4.61}$$

for $\beta = 1$ and by

$$\Sigma = \left[\begin{array}{c} \widehat{W}_{ij} \\ \hline w_{ia}^* \\ \hline \chi_{ib} \end{array}\right] = \left[\begin{array}{c} \widehat{W}_{ij} \\ \hline w_{ia}^* \\ \hline -\xi_{ia}^* \end{array}\right] \tag{4.62}$$

for $\beta = 2$, respectively. Here w_{ia} and ξ_{ia} are complex commuting and anti-commuting variables for $i = 1,\ldots,2N$, $a = 1,\ldots,\bar{n}$. As a consequence of the structure of Σ, see Eqs. (4.61) and (4.62), we find that

$$\Sigma^\dagger\Sigma = \widehat{W}^\dagger\widehat{W} + \chi^\dagger\chi + \left\{ \begin{array}{ll} w^\dagger w + w^T w^* & ,\beta = 1 \\ w^\dagger w & ,\beta = 2 \end{array}\right. \tag{4.63}$$

is a $\bar{n}\times\bar{n}$ dimensional matrix with commuting entries. The trace of $\widehat{W}\widehat{W}^\dagger AA^\dagger$ in the exponent of Eq. (4.59) can be written as supertrace of $\Sigma\Sigma^\dagger$ times a $(2N+p|2N+p)\times(2N+p|2N+p)$ supermatrix which is everywhere zero except in the upper left $p\times p$ block. The latter is given by $AA^\dagger$ such that

$$\mathrm{tr}\widehat{W}\widehat{W}^\dagger AA^\dagger = \mathrm{str}\Sigma\Sigma^\dagger \left[\begin{array}{cc} AA^\dagger & 0 \\ 0 & 0 \end{array}\right]\ . \tag{4.64}$$

We introduce $F(\Sigma^\dagger\Sigma)$ to be a function satisfying the integral theorem (4.60) for a given function f, it is not uniquely defined. However, the $\widehat{W}$ average is independent of the choice of F, it depends only on the function f. Hence, if we apply the integral theorem (4.60) from the right to the left with $f(\widehat{W}^\dagger\widehat{W}) = \mathrm{det}^{N+\tau\beta/2}\left(\widehat{W}^\dagger\widehat{W} + \mathbf{1}_{\bar{n}}\right)$, we obtain after an exchange of the Σ and A integration

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = {} & K_{n,p} t^{np\beta/2} \lim_{N\to\infty} K_N \int \mathrm{d}[A]\ \exp\left(\imath\mathrm{tr}\left(\mathbf{1}_p - \frac{t\beta}{2N}\Lambda^{-1}\right)AA^\dagger\right) \\ & \times \int \mathrm{d}[\Sigma]\ \frac{\exp\left(\imath\mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc} AA^\dagger & 0 \\ 0 & 0 \end{array}\right]\right)}{\mathrm{det}^{N+\tau\beta/2}\left(\Sigma^\dagger\Sigma + \mathbf{1}_{\bar{n}}\right)}\ . \end{aligned} \tag{4.65}$$

Due to this exchange of the integrals, the Σ integral turns into the characteristic function with respect to the distribution

$$P(\Sigma) \sim \det{}^{N+\tau\beta/2}\left(\Sigma^\dagger\Sigma + \mathbf{1}_{\bar{n}}\right) \tag{4.66}$$

on the space of rectangular supermatrices. It is a Fourier transformation and given by

$$\phi\left(\left[\begin{array}{cc} AA^\dagger & 0 \\ 0 & 0 \end{array}\right]\right) \sim \int d[\Sigma]\ \frac{\exp\left(\imath\,\mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc} AA^\dagger & 0 \\ 0 & 0 \end{array}\right]\right)}{\det{}^{N+\tau\beta/2}\left(\Sigma^\dagger\Sigma + \mathbf{1}_{\bar{n}}\right)}\ . \tag{4.67}$$

The supermatrix in the argument of ϕ is $(2N+p|2N+p)\times(2N+p|2N+p)$-dimensional. We assume that Υ is a supermatrix of the same size and symmetry class as $\Sigma\Sigma^\dagger$. The invariance of the Lorentzian probability distribution (4.66), $P(u\Sigma) = P(\Sigma)$ where u is in the proper supergroup, carries over to the characteristic function $\phi(\Upsilon)$ such that it is invariant under the adjoint action $\phi(u\Upsilon u^\dagger) = \phi(\Upsilon)$. Accordingly, the characteristic function depends only on the invariants $\mathrm{str}\Upsilon^m$, with $m \in \mathbb{N}$. With the aid of the trace duality (2.110), we find

$$\begin{aligned} \phi\left(\left[\begin{array}{cc} AA^\dagger & 0_{p\times 4N/\beta} \\ 0_{4N/\beta\times p} & 0_{4N/\beta\times 4N/\beta} \end{array}\right]\right) &= \phi\left(\left[\begin{array}{cc} 0_{(p+2N/\beta)\times(p+2N/\beta)} & 0_{(p+2N/\beta)\times 2N/\beta} \\ 0_{2N/\beta\times(p+2N/\beta)} & A^\dagger A \end{array}\right]\right) \\ &\sim \int d[\Sigma]\ \frac{\exp\left(\imath\,\mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc} 0 & 0 \\ 0 & A^\dagger A \end{array}\right]\right)}{\det{}^{N+\tau\beta/2}\left(\Sigma^\dagger\Sigma + \mathbf{1}_{\bar{n}}\right)}\ , \end{aligned} \tag{4.68}$$

where $AA^\dagger$ is a $2N/\beta \times 2N/\beta$ dimensional matrix with commuting entries. Analogous to the invariant matrix models constructed in section 4.1.2, it is either real quaternion self-dual corresponding to $\beta = 1$ or Hermitian to $\beta = 2$. If we insert

$$\mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc} 0 & 0 \\ 0 & A^\dagger A \end{array}\right] = -\mathrm{tr}\chi\chi^\dagger A^\dagger A \tag{4.69}$$

into Eq. (4.68) the entire $\widehat{W}$ and w-dependence of the integrand is concentrated in the determinant.

We make use of the identity (4.68) and replace the Σ integral in Eq. (4.65) by the right hand side of Eq. (4.68). Exchanging the A and the $\widehat{W}$, w integrals, reduces the A integral to a Gaussian. We perform it and are left with a dyadic $2N/\beta\times 2N/\beta$ dimensional matrix model,

$$\begin{aligned} E_p^{(\beta)}([0,t];p) &= K_{n,p}t^{np\beta/2}\lim_{N\to\infty} K_N \int d[\chi]\ Q(\chi^\dagger\chi) \\ &\times \det{}^{\beta/2}\left(\left(\mathbf{1}_p - \frac{t\beta}{2N}\Lambda^{-1}\right)\otimes\mathbf{1}_{2N/\beta} + \mathbf{1}_p\otimes\chi\chi^\dagger\right)\ , \end{aligned} \tag{4.70}$$

where we introduce the distribution function [140]

$$
\begin{aligned}
Q(\chi^\dagger\chi) &= \int \mathrm{d}[\widehat{W}, w]\ \det{}^{-(N+\tau\beta/2)}\left(\Sigma^\dagger\Sigma + \mathbf{1}_{\bar{n}}\right) \\
&\sim \det{}^{-\beta(n+2/\beta-1)/2}\left(\chi^\dagger\chi + \mathbf{1}_{\bar{n}}\right) \ .
\end{aligned}
\tag{4.71}
$$

Due to the dyadic structure, we have two possibilities to analyze Eq. (4.70) both using the generalized Hubbard-Stratonovich transformation. Either we replace the $2N/\beta \times 2N/\beta$ dimensional matrix $\chi\chi^\dagger$ or the $\bar{n} \times \bar{n}$ dimensional matrix $\chi^\dagger\chi$. For $\beta = 2$ both matrices are Hermitian. For $\beta = 1$ is $\chi\chi^\dagger$ real quaternion self-dual, but $\chi^\dagger\chi$ is real symmetric. Thus, replacing the former has the advantage, that we study an infinite dimensional matrix model that possesses for $\beta = 1$ a twofold degeneracy in the eigenvalues so that all square roots in Eq. (4.70) become integer powers. Whereas, if we replace $\chi\chi^\dagger$, we obtain a finite dimensional matrix model such that taking the $N \to \infty$ limit does not vary the matrix size. The price of the latter case is that for $\beta = 1$ square roots do remain in the integrand such that analytically analyzing it further is impossible.

4.2.3 Exact Expression for $\beta = 2$

For the complex correlated Wishart model, the matrix $\chi^\dagger\chi$ is Hermitian. We apply the generalized Hubbard-Stratonovich transformation, replace it by an ordinary Hermitian matrix h and find

$$
\begin{aligned}
E_p^{(2)}([0,t];p) = K_{n,p}t^{np} \lim_{N\to\infty} K_N \det{}^{N-p}&\left(\mathbf{1}_p - \frac{t}{N}\Lambda^{-1}\right) \\
\times \int \mathrm{d}[h]\ &\det{}^{-n}\left(h + \mathbf{1}_p\right) f_{N,2}(h) \\
\times \prod_{k=1}^{p} &\det{}^{-1}\left(\left(1 - \frac{t}{N}\Lambda_k^{-1}\right)\mathbf{1}_p + h\right) \ .
\end{aligned}
\tag{4.72}
$$

The function $f_{N,2}(h)$ was introduced in section 4.1.2 and is given by Eq. (4.15). It is a distribution on the space of Hermitian matrices and depends on the eigenvalues of h only, *i.e.* $f_{N,2}(h) = f_{N,2}(s)$ with $h = UsU^\dagger$ and $U \in \mathrm{U}(p)$. In terms of the eigenvalues s_i it reduces to Eq. (4.23). Thus, diagonalizing h and substituting it into the resulting expression yields

$$
\begin{aligned}
E_p^{(2)}([0,t];p) = K_{n,p}t^{np} \lim_{N\to\infty} K_N &\det{}^{N-p}\left(\mathbf{1}_p - \frac{t}{N}\Lambda^{-1}\right) \\
\times \int \mathrm{d}[s]\ \Delta_p^2(s) &\prod_{i=1}^{p} \frac{\delta^{(N+p-1)}(s_i)}{(s_i+1)^n \prod_{k=1}^{p}\left(1 - t\Lambda_k^{-1}/N + s_i\right)} \ ,
\end{aligned}
\tag{4.73}
$$

with $\delta^{(m)}(x)$ the mth derivative of the δ-function. Because of the Vandermonde determinant in Eq. (4.73), the $N \to \infty$ limit has to be taken carefully. To this end, the product of determinants in the denominator is combined with a Vandermonde determinant [111,112]

$$\frac{\Delta_p(s)}{\prod_{i,k}^{p,p}\left(1-t\Lambda_k^{-1}/N-s_i\right)} \sim \frac{\det^{p-1}\Lambda}{t^{p(p-1)/2}\Delta_p(\Lambda)}\det\left[\frac{1}{1-t\Lambda_k^{-1}/N+s_i}\right] , \tag{4.74}$$

where $1 \leq i,k \leq p$. Similarly, we bring together the remaining Vandermonde determinant and the weight,

$$\frac{\Delta_p(s)\prod_{i=1}^p \delta^{(N+p-1)}(s_i)}{\prod_{i=1}^p (s_i+1)^n} \sim \det\left[\frac{s_i^{j-1}\delta^{(N+p-1)}(s_i)}{(1+s_i)^n}\right] , \tag{4.75}$$

with $1 \leq i,j \leq p$. If we substitute Eqs. (4.74) and (4.75) into the eigenvalue integral (4.73), we arrive at

$$\begin{aligned} E_p^{(2)}([0,t];p) = {}& \frac{K_{n,p}t^{np}}{t^{p(p-1)/2}\Delta_p(\Lambda)}\lim_{N\to\infty} K_N \mathrm{det}^{N-p}\left(\mathbf{1}_p - \frac{t}{N}\Lambda^{-1}\right) \\ & \times \int \mathrm{d}[s]\ \det\left[\frac{s_i^{j-1}\delta^{(N+p-1)}(s_i)}{(1+s_i)^n}\right]\det\left[\frac{1}{1-t\Lambda_k^{-1}/N+s_i}\right] . \end{aligned} \tag{4.76}$$

The columns in the determinants depend on one of the p eigenvalues s_i only. This kind of integral is covered by Andreijefs integral theorem, see Ref. [111] and references therein. Applying it to the gap probability (4.76) leads to

$$E_p^{(2)}([0,t];p) = \frac{K_{n,p}t^{np-p(p-1)/2}}{\Delta_p(\Lambda)}\lim_{N\to\infty} K_N \mathrm{det}^{N-p}\left(\mathbf{1}_p - \frac{t}{N}\Lambda^{-1}\right)\det\left[W_{jk}\right] , \tag{4.77}$$

where $1 \leq j,k \leq p$ and we introduce the determinant kernel as

$$W_{jk} = \int_{-\infty}^{\infty} \mathrm{d}z \frac{z^{j-1}\delta^{(N+p-1)}(z)}{(1+z)^n(1-t\Lambda_k^{-1}/N+z)} . \tag{4.78}$$

Hence, we reduce the computation of the eigenvalue integral to the calculation of the matrix kernel. We perform $N+p-1$ partial integrations and bring the derivatives of the δ-function to the remaining integrand. Because of the latter, all boundary terms vanish such that W_{ik} is given by the $N+p-1$ fold derivative of the integrand (4.78), evaluated at zero. The derivatives are done by applying partial fraction decomposition to the denominator in Eq. (4.78),

$$\frac{1}{(1+z)^n(1-t\Lambda_k^{-1}/N+z)} = \frac{(\lambda_k N/t)^n}{1-t\Lambda_k^{-1}/N+z} - \sum_{l=1}^n \frac{(\Lambda_k N/t)^{n-l+1}}{(1+z)^l} . \tag{4.79}$$

such that by substituting this into the expression (4.78), we obtain

$$\begin{aligned} W_{jk} = {}& (-1)^{j-1}(j-1)!(N+p-j)! \\ & \times \left(\frac{(\Lambda_k N/t)^n}{\left(1-t\Lambda_k^{-1}/N\right)^{N+p-j+1}} - \sum_{l=1}^n \frac{(\Lambda_k N/t)^{n-l+1}\,(N+p-j+l-1)!}{(l-1)!(N+p-j)!}\right) . \end{aligned} \tag{4.80}$$

For later purpose, we introduce $W_{jk}^{(0)}$ as the second row of above expression. If we perform the $N \to \infty$ limit, we have to respect the anti-symmetrizing property of the determinant. Because, carelessly taking the $N \to \infty$ limit in each row individually results in a matrix where all rows are the same such that the determinant of this matrix is zero. Hence, in order to respect the anit-symmetrizing property of the determinant in Eq. (4.77), we expand each row in powers of $1/N$ and build linear combinations until all rows are linearly independent for $N \to \infty$. To do so, we have to expand the first summand of $W_{jk}^{(0)}$ only,

$$\frac{1}{\left(1 - t\Lambda_k^{-1}/N\right)^{N+p-j+1}} = \frac{1}{\left(1 - t\Lambda_k^{-1}/N\right)^{N}} \sum_{m=0}^{\infty} \left(\frac{t}{N\Lambda_k}\right)^m \frac{(p-j+m)!}{(p-j)!} \ . \tag{4.81}$$

If we insert Eq. (4.81) into $W_{jk}^{(0)}$ and subtract $W_{(j+1)k}^{(0)}$ from $W_{jk}^{(0)}$ for $j = 1, \ldots, p-1$, we find as new jth element

$$\begin{aligned} W_{jk}^{(1)} &= W_{jk}^{(0)} - W_{(j+1)k}^{(0)} \\ &= \frac{(\Lambda_k N/t)^{n-1}}{\left(1 - t\Lambda_k^{-1}/N\right)^{N}} \sum_{m=0}^{\infty} \left(\frac{t}{N\Lambda_k}\right)^m \frac{(p-j+m)!(m+1)!}{(p-j)!m!} \\ &- \left(\frac{t}{\Lambda_k N}\right) \sum_{l=1}^{n-1} \frac{(\Lambda_k N/t)^{n-l+1} (N+p-j+l-1)!}{(l-1)!(N+p-j)!} \ , \end{aligned} \tag{4.82}$$

where the superscript (1) indicates the first step of an iteration. The leading order in N of $W_{jk}^{(0)}$ and $W_{jk}^{(1)}$ are N^n and N^{n-1}, respectively. Thus, by the subtraction of the $(j+1)$th row from the jth row, we decrease the order of N in the resulting row by one. We repeat this procedure iteratively in i, and subtract in each iteration $W_{(j+1)k}^{(i)}$ from $W_{jk}^{(i)}$ for all $j = 1, \ldots, p-i-1$, with

$$\begin{aligned} W_{jk}^{(i)} &= \frac{(\Lambda_k N/t)^{n-i}}{\left(1 - t\Lambda_k^{-1}/N\right)^{N}} \sum_{m=0}^{\infty} \left(\frac{t}{N\Lambda_k}\right)^m \frac{(p-j+m)!(m+i)!}{(p-j)!m!} \\ &- \left(\frac{t}{\Lambda_k N}\right)^i \sum_{l=1}^{n-i} \frac{(\Lambda_k N/t)^{n-l+1} (N+p-j+l-1)!}{(l-1)!(N+p-j)!} \ , \end{aligned} \tag{4.83}$$

until i reaches $p-1$. The resulting set of row vectors is linearly independent, with different degrees in N. According to the determinant in Eq. (4.79) and its invariance under building row or column wise linear combinations, we replace W_{jk} by $W_{jk}^{(p-j)}$. We absorb all constant factors and powers of N in K, respectively K_N such that Eq. (4.79) becomes

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{K_{n,p} t^{np-p(p-1)/2}}{\Delta_p(\Lambda)} \\ &\times \lim_{N\to\infty} K_N \mathrm{det}^{N-p} \left(\mathbf{1}_p - \frac{t}{N}\Lambda^{-1}\right) \det\left[W_{jk}^{(p-j)}\right] \ . \end{aligned} \tag{4.84}$$

All rows in the determinant above are linear independent in the $N \to \infty$ limit. Hence, we can perform the limit $N \to \infty$ by keeping only the leading order term in N of each individual row. The leading power in N is again absorbed into K_N such that we arrive at

$$E_p^{(2)}([0,t];p) = \frac{t^{np-p(p-1)/2}(-1)^{p(p-1)/2}}{\det^{n-p+1}\Lambda\ \Delta_p(\Lambda)} \times \det\left[\left(\frac{\Lambda_k}{t}\right)^{n-p+j} - \sum_{l=1}^{n-p+j} \frac{\exp\left(-t\Lambda_k^{-1}\right)}{(p-j)!(l-1)!}\left(\frac{\Lambda_k}{t}\right)^{n-p+j+1-l}\right], \tag{4.85}$$

where $1 \leq j,k \leq p$. The last step from Eq. (4.84) to Eq. (4.85) is justified, because the normalization constant K_N is derived by requiring the $N \to \infty$ limit to be finite, whereas $K_{n,p}$ is determined by $E_p^{(2)}([0,t];p) \to 1$ for $t \to \infty$. As we will see in section 5.2.3, the gap probability (4.85) is similar to the expression we find when using the unitary Itzykson-Zuber integral to compute Eq. (2.37).

To confirm our findings we compare them to numerical simulations. We generate a sample of $25\ 000$ complex Wishart correlation matrices consisting of 3×5 dimensional sample data matrices and choose the empirical eigenvalues to be $\Lambda_k = 5, 3, 1$. We find a perfect agreement between the analytic expression and the numerical simulations shown in Fig. 4.5.

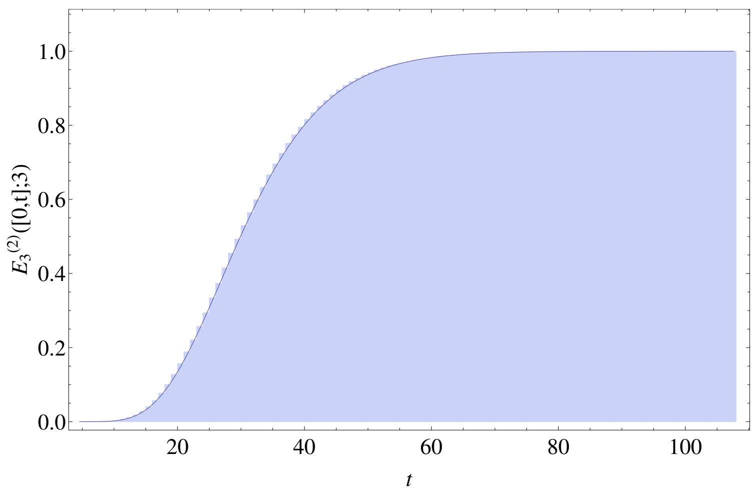


Figure 4.5: Comparison of the analytic expression (line) with a numerical simulation (histogram) of the gap probability $E_p^{(2)}([0,t];p)$ with $p = 3,\ n = 5$ and $\Lambda_k = 5, 3, 1$.

4.3 Eigenvalue Statistics in the Correlated Jacobi Ensemble

The correlated Jacobi model as introduced in section 2.2.3 is an ensemble of Hermitian matrices $\mathcal{H}$ composed of two independent, correlated Wishart matrices $FF^\dagger$ and $BB^\dagger$,

$$\mathcal{H} = \frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger} \tag{4.86}$$

The matrices F and B have the same number of rows p, but a different number of columns n_1, respectively, n_2. They are Gaussian distributed according to Eq. (2.16) with respect to different empirical correlation matrices C_F and C_B. If $C_F = C_B = C$, $\mathcal{H}$ becomes independent of the empirical correlation matrix C [41]. However, whenever $C_F \neq C_B$ the eigenvalue statistics are highly non-trivial, even for $\beta = 2$.

The calculation shown in this section is based on Ref. [7] and extends some aspects of the calculations therein in more detail.

We present two different approaches to consider eigenvalue statistics. The first approach, described in section 4.3.1, applies the results of section 2.5 to the case of a two-matrix model. It leads to an average over two independent supermatrices. In section 4.3.2, we derive an equivalent description of the eigenvalue statistics of $\mathcal{H}$ in terms of a Lorentzian distributed matrix ensemble and map it to a supermatrix model. We solve it separately for the complex and the real case. The former is considered in section 4.3.3. The latter is solved in section 4.3.4.

4.3.1 Two Supermatrix Model

In section 2.3.1, we show how to study eigenvalue statistics of Hermitian matrices in terms of averaged ratios of characteristic polynomials. We replace in Eq. (2.35), H by $\mathcal{H}$ and analyze the generating function

$$Z_p^{k/k}(\kappa) = K \int \mathrm{d}[F,B]\ \frac{\prod_{a=1}^{k} \det\left(\mathcal{H} - \kappa_{a2}\mathbf{1}_{\gamma_2 p}\right)}{\prod_{b=1}^{k} \det\left(\mathcal{H} - \kappa_{b1}\mathbf{1}_{\gamma_2 p}\right)} P(F|C_F)P(B|C_B)\ . \tag{4.87}$$

to study the k-point correlation function. The normalization constant K is determined by $Z_p^{k/k}(\kappa \to 0) \to 1$. In appendix B.4 we show that because of the invariance of the flat measure $\mathrm{d}[F,B]$, the generating function depends only on the always non-negative eigenvalues of $C_B C_F^{-1}$. We ordered them in $\hat{\Lambda}_{\text{eff}} = \mathbf{1}_{\gamma_2} \otimes \Lambda_{\text{eff}}$ where $\Lambda_{\text{eff}} = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_p)$. Employing this observation in Eq. (4.87) corresponds to $C_F \mapsto \mathbf{1}_{\gamma_2 p}$ and $C_B \mapsto \hat{\Lambda}_{\text{eff}}$.

To apply the analysis of section 2.5 in the following to Eq. (4.87), the arguments of the characteristic polynomials have to be linear in $FF^\dagger$ and $BB^\dagger$. This is achieved by taking $(FF^\dagger + BB^\dagger)^{-1}$ out of the ratio of determinants using

$$\frac{\det\left(\frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger} - \kappa_{a2}\mathbf{1}_{\gamma_2 p}\right)}{\det\left(\frac{FF^\dagger - BB^\dagger}{FF^\dagger + BB^\dagger} - \kappa_{b1}\mathbf{1}_{\gamma_2 p}\right)} = \left(\frac{1+\kappa_{a2}}{1+\kappa_{b1}}\right)^p \frac{\det\left(FF^\dagger \frac{1-\kappa_{a2}}{1+\kappa_{a2}} - BB^\dagger\right)}{\det\left(FF^\dagger \frac{1-\kappa_{b1}}{1+\kappa_{b1}} - BB^\dagger\right)}\ . \tag{4.88}$$

For each ratio of determinants in Eq. (4.87) we substitute Eq. (4.88) such that we arrive at

$$\begin{aligned} Z_p^{k/k}(\kappa) = K\text{sdet}^{-p}\left(\mathbf{1}_{k|k}+\kappa\right)\int \text{d}[F,B]\ P(F|\mathbf{1}_{\gamma_2 p})P(B|\hat{\Lambda}_{\text{eff}}) \\ \times \prod_{a=1}^{k}\frac{\det\left(FF^\dagger\frac{1-\kappa_{a2}}{1+\kappa_{a2}}-BB^\dagger\right)}{\det\left(FF^\dagger\frac{1-\kappa_{a1}}{1+\kappa_{a1}}-BB^\dagger\right)}\ , \end{aligned} \tag{4.89}$$

where $\kappa = \text{diag}(\kappa_{11},\ldots,\kappa_{k1},\kappa_{12},\ldots,\kappa_{k2})$. The application of the results of section 2.5 to the generating function (4.89) is straightforward and is shown in appendix B.5 in more detail. There we construct a two-supermatrix model dual to averaged product of ratios of characteristic polynomials (4.89) given by

$$\begin{aligned} Z_p^{k/k}(\kappa) = K\text{sdet}^{-p}\left(\mathbf{1}_{k|k}+\kappa\right)\int \text{d}[\sigma]\ \text{d}[\varrho]\ I_{n_2}(\varrho)I_{n_1}(\sigma)\exp\left(-\text{str}\varrho-\text{str}\sigma\right) \\ \times\ \text{sdet}^{-1/\tilde{\gamma}}\left(\mathbf{1}_p\otimes\sigma-\Lambda_{\text{eff}}\otimes\varrho\mathbf{j}\right)\ , \end{aligned} \tag{4.90}$$

where the function I_{n_i} is the supersymmetric Ingham-Siegel integral (2.97), ρ and σ are as explained in section 2.4.3 and

$$\mathbf{j} = \mathbf{1}_{\tilde{\gamma}}\otimes\text{diag}\left(\frac{1-\kappa_{1,1}}{1+\kappa_{1,1}},\ldots,\frac{1-\kappa_{k,1}}{1+\kappa_{k,1}},\frac{1-\kappa_{1,2}}{1+\kappa_{1,2}},\ldots,\frac{1-\kappa_{k,2}}{1+\kappa_{k,2}}\right)\ . \tag{4.91}$$

The generating function (4.90) for the density, *i.e.* for $k=1$, could be computed by expanding the integrand in the anticommuting variables and performing the remaining integrals. Because of the two supermatrix integrals, this is a non-trivial task for the complex correlated Jacobi ensemble. Thus, in the next section, we use more advanced techniques to show that the generating function (4.87) has a dual supermatrix model consisting of one supermatrix only.

4.3.2 Lorentzian Supermatrix Model

We apply the projection formula as introduced in Ref. [139,140] to derive a Lorentzian distributed supermatrix model. To do so, we use the fact that the generating function (4.87) has dual representation as an average over a Lorentzian matrix model.

For reasons of clarity, we express in the following the product of ratios of characteristic polynomials as superdeterminant and consider

$$\begin{aligned} Z_p^{k/k}(\kappa) = K\int \text{d}[F,B]\ P(F|\mathbf{1}_{p\gamma_2})P(B|\hat{\Lambda}_{\text{eff}}) \\ \times\ \text{sdet}^{-1}\left[\frac{FF^\dagger-BB^\dagger}{FF^\dagger+BB^\dagger}\otimes\mathbf{1}_{k|k}-\mathbf{1}_{\gamma_2 p}\otimes\kappa\right] \end{aligned} \tag{4.92}$$

The equivalence of the two-Wishart matrix average (4.92) and a generating function with Lorentzian distribution is shown in appendix B.6 by a series of coordinate changes and an average over the F ensemble. The expression obtained thereby depends by construction on a Lorentzian distributed matrix $BB^\dagger$ only and reads

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int d[B] \; \text{sdet}^{-1} \left[\frac{\hat{\Lambda}_{eff}^{-1} - BB^\dagger}{\hat{\Lambda}_{eff}^{-1} + BB^\dagger} \otimes \mathbf{1}_{k|k} - \mathbf{1}_{\gamma_2 p} \otimes \kappa \right] \\ \times \det{}^{-n/\gamma_1} \left(B^\dagger B + \mathbf{1}_{\gamma_2 n_2} \right) \; , \end{aligned} \tag{4.93}$$

where $n = n_1 + n_2$. Although the superdeterminant in Eq. (4.93) is not linear in $BB^\dagger$, it is ideal to apply the ideas of Refs. [139,140] and map it to superspace. To begin, we linearize the argument of the superdeterminant in $BB^\dagger$ using the identity (4.88) and find

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int d[B] \; \text{sdet}^{-1} \left[\hat{\Lambda}_{\text{eff}}^{-1} \otimes \frac{\mathbf{1}_{k|k} - \kappa}{\mathbf{1}_{k|k} + \kappa} - BB^\dagger \otimes \mathbf{1}_{k|k} \right] \\ \times \det{}^{-n/\gamma_1} \left(B^\dagger B + \mathbf{1}_{\gamma_2 n_2} \right) \; . \end{aligned} \tag{4.94}$$

We thereby obtain a matrix model similar to the one considered in section 4.2.2. There we mapped it to an ordinary matrix model with respect to a Lorentzian distribution. We adapt this construction in appendix B.7 to the present case and we arrive at a representation of the generating function (4.87) in terms of an average over a $(k\tilde{\gamma}|k\tilde{\gamma}) \times (k\tilde{\gamma}|k\tilde{\gamma})$-dimensional supermatrix $\chi\chi^\dagger$,

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int d[\chi] \; \text{sdet}^{-(n-p)/\gamma_1} \left(\chi\chi^\dagger + \mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k} \right) \\ \times \text{sdet}^{-1/\gamma_1} \left(\Lambda_{\text{eff}}^{-1} \otimes (\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k} - \hat{\kappa}) - \mathbf{1}_p \otimes (\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k} + \hat{\kappa})\chi\chi^\dagger \right) \; , \end{aligned} \tag{4.95}$$

where $\hat{\kappa} = \text{diag}(\mathbf{1}_{\tilde{\gamma}} \otimes \kappa_1, \mathbf{1}_{\tilde{\gamma}} \otimes \kappa_2)$. In the next sections, we employ for $k = 1$ the generalized Hubbard-Stratonovich transformation and replace in Eq. (4.95) the matrix $\chi\chi^\dagger$ by a supermatrix.

4.3.3 Eigenvalue Density in the Complex Ensemble

To compute the eigenvalue distribution in the complex correlated Jacobi ensemble, we use Eq. (4.95), set $k = 1$ and apply the generalized Hubbard-Stratonovich transformation, see section 2.4.3, yielding

$$\begin{aligned} Z_p^{1/1}(\kappa) = K \int d[\sigma] \; \text{sdet}^{-(n-p)} \left(\sigma + \mathbf{1}_{1|1} \right) I_{n_1}(\sigma) \\ \times \prod_{k=1}^{p} \text{sdet}^{-1} \left((\mathbf{1}_{1|1} - \hat{\kappa})/\Lambda_k - (\mathbf{1}_{1|1} + \hat{\kappa})\sigma \right) \; , \end{aligned} \tag{4.96}$$

where $I_{n_1}(\sigma)$ is the supersymmetric Ingham-Siegel integral (2.97). The domain of integration in Eq. (4.96) is determined by the symmetries of $\chi\chi^\dagger$ and therefore given by the space of Hermitian supermatrices,

$$\sigma = \begin{bmatrix} a & \eta \\ -\eta^* & \imath b \end{bmatrix} \tag{4.97}$$

where $a, b \in \mathbb{R}$ and η, η^* are Grassmannians. The normalization constant K in Eq. (4.97) is fixed by the condition that $Z_p^{1/1}(\kappa \to 0) \to 1$, see also Eq. (4.87). Applying it to Eq. (4.96), we obtain an invariant Hermitian supermatrix model. These kind of integrals are covered by integral theorems, *c.f.* Refs. [65,152]. To apply them to the resulting expression, we specify the integration measure. We choose it to be $\mathrm{d}[\sigma] = \mathrm{d}a\mathrm{d}b\mathrm{d}\eta\mathrm{d}\eta^*$ and find after application of the integral theorems that

$$K = -1 \; . \tag{4.98}$$

We insert Eq. (4.98) into Eq. (4.96) and derive the eigenvalue density of the correlated Jacobi ensemble from the generating function (4.96). To do so we set $\kappa_{11} = x \pm \imath\varepsilon$ and $\kappa_{12} = x_1$ and use Eq. (2.34), yielding

$$S(x) = \frac{1}{2\pi\imath p} \lim_{\varepsilon \to 0^+} \frac{\partial}{\partial x_1}\bigg|_{x_1 = x} \left(Z_p^{1/1}(x_1, x + \imath\varepsilon) - Z_p^{1/1}(x_1, x - \imath\varepsilon) \right) \; , \tag{4.99}$$

where $S(x)$ is normalized such that $\int_0^\infty \mathrm{d}x S(x) = 1$. The supersymmetric Ingham-Siegel integral, I_{n_1}, restricts the boson-boson block of σ to lie on the positive real line and the fermion-fermion block of σ to a kind of contour integral. Following the analysis of Ref. [163], it reduces to elementary integrals and is given by

$$I_{n_1}(\sigma) = \frac{(-\imath) 2\pi \Theta(a)(\imath a)^{n_1 - 1}}{(n-1)!} \frac{\partial^{n_1 - 1}}{\partial b^{n_1 - 1}} \left(\eta^* \eta \frac{\partial}{\partial b} + \imath a \right) \delta(b) \; . \tag{4.100}$$

The derivatives of the δ-function should be interpreted as derivatives of the remaining integrand with respect to b, evaluated at $b = 0$.

To perform the η, η^* integrals in Eq. (4.96), we first expand each part of the integrand (4.96) separately in η and η^* and multiply them afterwards. We begin with the determinant in the first line of Eq. (4.96) and find

$$\mathrm{sdet}^{-(n-p)}(\sigma + \mathbf{1}_2) = \left(\frac{\imath b + 1}{a + 1} \right)^{n-p} \left(1 + (n-p) \frac{\eta^* \eta}{(\imath b + 1)(a + 1)} \right) \; . \tag{4.101}$$

A similar expansion is found for the product of superdeterminants running over the empirical eigenvalues. It reads

$$\begin{aligned} \prod_{k=1}^{p} \mathrm{sdet}^{-1}\left((\mathbf{1}_2 - \hat{\kappa})/\Lambda_k - (\mathbf{1}_2 + \hat{\kappa})\sigma \right) = \prod_{k=1}^{p} \frac{(1 - \kappa_{12})/\Lambda_k - (1 + \kappa_{12})\imath b}{(1 - \kappa_{11})/\Lambda_k - (1 + \kappa_{11})a} \\ \times \left(1 + \sum_{k=1}^{p} \frac{(1 + \kappa_{11})(1 + \kappa_{12})\eta^* \eta}{((1 - \kappa_{12})/\Lambda_k - (1 + \kappa_{12})\imath b)((1 - \kappa_{11})/\Lambda_{k'} - (1 + \kappa_{11})a)} \right) \; . \end{aligned} \tag{4.102}$$

If we substitute Eq. (4.100), Eq. (4.101) and Eq. (4.102) into the generating function (4.96) and do the Berezinian integrals over η, η^*, we are left with a twofold integral,

$$
\begin{aligned}
Z_p^{1/1}(\kappa) = {} & \frac{\imath}{(n_1-1)!}\int\limits_0^\infty \mathrm{d}a \int\limits_{-\infty}^\infty \mathrm{d}b (\imath a)^{n_1-1}\left(\frac{\imath b+1}{a+1}\right)^{n-p} \\
& \times \prod_{k=1}^p \frac{(1-\kappa_{12})/\Lambda_k-(1+\kappa_{12})\imath b}{(1-\kappa_{11})/\Lambda_k-(1+\kappa_{11})a}\left(\frac{\partial^{n_1}}{\partial b^{n_1}}+\frac{\imath a(n-p)}{(\imath b+1)(a+1)}\frac{\partial^{n_1-1}}{\partial b^{n_1-1}}\right. \\
& \left.+\sum_{k=1}^p \frac{(1+\kappa_{11})(1+\kappa_{12})\imath a}{((1-\kappa_{12})/\Lambda_k-(1+\kappa_{12})\imath b)((1-\kappa_{11})/\Lambda_k-(1+\kappa_{11})a)}\frac{\partial^{n_1-1}}{\partial b^{n_1-1}}\right)\delta(b)\ .
\end{aligned}
\tag{4.103}
$$

The derivatives of the δ-function are mapped by partial integration to the remaining integrand. All boundary terms arising due to the partial integration are zero. Because of the p-fold product, the differentiation with respect to b is not straightforward. To simplify its evaluation, we use the elementary-symmetric functions (4.25) and the identity

$$
\prod_{i=1}^p (x-\alpha_i) = \sum_{i=0}^p (-1)^i e_i(\alpha) x^{p-i}\ . \tag{4.104}
$$

Inserting the sum (4.104) into Eq. (4.105) reduces the differentiation of the integrand with respect to b to the differentiation of a polynomial in $\imath b+1$ and $\imath b$. By setting $\kappa_{11}=x_0=x\pm\imath\varepsilon$ and $\kappa_{12}=x_1$, the generating function becomes

$$
\begin{aligned}
Z_p^{1/1}(x_1,x_0) = {} & \int\limits_0^\infty \mathrm{d}a \frac{a^{n_1-1}}{(a+1)^{n-p}}\prod_{k=1}^p \frac{1}{(1-x_0)/\Lambda_k-(1+x_0)a} \\
& \times\left(\sum_{k=0}^p \frac{e_{p-k}(\Lambda^{-1})(-1)^k(1-x_1)^{p-k}(1+x_1)^k(n-p)!}{(n_1-1-k)!(n_2-p+k)!}\left(\frac{n_1}{n_1-k}-\frac{a}{a+1}\right)\right. \\
& \left.-\sum_{l=1}^p\sum_{k=0}^{p-1}\frac{e_{p-k-1}((\Lambda^{\hat{l}})^{-1})(-1)^k(1-x_1)^{p-k-1}(1+x_0)(1+x_1)^{k+1}(n-p)!a}{(n_1-1-k)!(n_2-p+k+1)!((1-x_0)/\Lambda_l-(1+x_0)a)}\right)\ ,
\end{aligned}
\tag{4.105}
$$

where we introduce $\Lambda^{\hat{l}}$ as the $p-1$ dimensional diagonal matrix consisting of all entries of Λ except the lth.

Before we consider the a integral in detail, we perform the differentiation with respect to x_1 at $x_1=x$, as explained in Eq. (4.99). This corresponds in expression (4.105) to a substitution of the form

$$
\begin{aligned}
(1-x_1)^{p-k}(1+x_1)^k &\mapsto (1-x)^{p-k-1}(1+x)^{k-1}(2k-p(1+x))\ , \\
(1-x_1)^{p-k-1}(1+x_1)^{k+1} &\mapsto (1-x)^{p-k-2}(1+x)^k(2(k+1)-p(1+x))\ .
\end{aligned}
$$

This derivative does not affect the structure of the a integral. It has poles in the complex plane, which for $\varepsilon\to 0$ lie on the real line. Thus, the integral becomes rather simple if we take $\varepsilon \to 0$ together with the imaginary part. Since, the entire ε dependence of the integrand (4.105) resides in the p-fold product running over the empirical eigenvalues, this leads to

$$\begin{aligned}
&\frac{1}{\pi}\mathrm{Im}\lim_{\varepsilon\to 0}\prod_{k=1}^{p}\frac{1}{(1-x_0)/\Lambda_k-(1+x_0)a} \\
&=-\frac{1}{1+x}\sum_{i=1}^{p}\delta\left(\frac{1-x}{1+x}\Lambda_i^{-1}-a\right)\prod_{\substack{k'=1\\k'\neq i}}^{p}\text{p.v.}\left[\frac{1}{\frac{1-x}{\Lambda_{k'}}-\frac{1-x}{\Lambda_i}}\right] ,
\end{aligned} \tag{4.106}$$

where p.v. means principal value. The equality sign in Eq. (4.106) holds under an integral only, because we use that products of δ-functions with different arguments are zero, *i.e.*

$$\int_{-\infty}^{\infty}\mathrm{d}x f(x)\delta(x-a)\delta(x-b)=0 , \tag{4.107}$$

for $a\neq b$. In the third line of Eq. (4.105) the p-fold product includes not only poles of order one but also one pole of order two such that taking the $\varepsilon \to 0$ limit and the imaginary part leads to

$$\begin{aligned}
&\frac{1}{\pi}\mathrm{Im}\lim_{\varepsilon\to 0}\frac{\partial}{\partial\alpha}\prod_{k=1}^{p}\frac{1}{(1-x_0)/\Lambda_k-(1+x_0)a-\alpha\delta_{lk}}\Bigg|_{\alpha=0}=-\frac{1}{1+x}\sum_{i=1}^{p}\frac{\partial}{\partial\alpha} \\
&\times\delta\left(\frac{1-x}{1+x}\Lambda_i^{-1}-a-\alpha\delta_{li}\right)\prod_{\substack{k=1\\k\neq i}}^{p}\text{p.v.}\left[\frac{1}{\frac{1-x}{\Lambda_k}-\frac{1-x}{\Lambda_i}-\alpha\delta_{lk}}\right]\Bigg|_{\alpha=0} .
\end{aligned} \tag{4.108}$$

Thus, the computation of the a integral reduces, because of the δ-functions to an evaluation of the integrand at particular values of a. If we substitute Eq. (4.106) and Eq. (4.108) into the generating function (4.105) and multiply it by $1/p$, we arrive at an exact expression for the level density in the complex correlated Jacobi ensemble

$$\begin{aligned}
S_2(x)=&\frac{-1}{p(1+x)}\sum_{i=1}^{p}\frac{\left(\frac{1-x}{1+x}\Lambda_i^{-1}\right)^{n_1-1}}{\left(1+\frac{1-x}{1+x}\Lambda_i^{-1}\right)^{n-p}\prod_{\substack{j=1\\j\neq i}}^{p}\left(\Lambda_i^{-1}-\Lambda_j^{-1}\right)} \\
&\times\left[\sum_{k=0}^{p}\frac{e_{p-k}(\Lambda^{-1})(-1)^k(1-x)^{-k}(1+x)^k(2k-p(1+x))(n-p)!}{(n_1-k-1)!(n_2-p+k)!}\right. \\
&\times\left(\frac{n_1}{n_1-k}-\frac{\frac{1-x}{1+x}\Lambda_i^{-1}}{1+\frac{1-x}{1+x}\Lambda_i^{-1}}\right)-\sum_{\substack{l=1\\l\neq i}}^{p}\sum_{k=0}^{p-1}\frac{e_{p-k-1}((\Lambda^{\hat{i}})^{-1})(-1)^k(n-p)!}{(n_1-k-1)!(n_2-p+k)!} \\
&\left.\times\frac{(1-x)^{-k-1}(1+x)^k(2(k+1)-p(1+x))}{\Lambda_i\left(\Lambda_l^{-1}-\Lambda_i^{-1}\right)}\right]
\end{aligned} \tag{4.109}$$

$$
\begin{aligned}
&+ \frac{1}{p(1+x)} \sum_{i=1}^{p} \sum_{k=0}^{p-1} \frac{e_{p-k-1}((\Lambda^{\hat{i}})^{-1})(-1)^k (n-p)!}{(n_1-k-1)!(n_2-p+k)!} \frac{\left(\frac{1-x}{1+x}\Lambda_i^{-1}\right)^{n_1-1}}{\left(1+\frac{1-x}{1+x}\Lambda_i^{-1}\right)^{n-p+1}} \\
&\times \frac{(1+x)^k(2(k+2)-p(1+x))}{(1-x)^{k+1}\prod_{\substack{j=1\\ j\neq i}}^{p}\left(\Lambda_i^{-1}-\Lambda_j^{-1}\right)} \left[(n-p)\frac{1-x}{1+x}\Lambda_i^{-1} - n_1\left(1+\frac{1-x}{1+x}\Lambda_i^{-1}\right)\right. \\
&\left. -(1-x)\Lambda_i^{-1}\left(1+\frac{1-x}{1+x}\Lambda_i^{-1}\right)\sum_{\substack{l=1\\ l\neq i}}^{p}\frac{1}{(1-x)\left(\Lambda_l^{-1}-\Lambda_i^{-1}\right)}\right] .
\end{aligned}
$$

Although Eq. (4.109) looks quite complex, it can be evaluated using MATHEMATICA [137] within a few seconds. In Fig. 4.6 we compare the density (4.109) with Monte-Carlo simulations of a complex correlated Jacobi matrix. For the comparison we take $p = 3, n_1 = 5$ and $n_2 = 7$ and chose the empirical eigenvalues to be $\Lambda = \text{diag}(1/3, 2, 4.5)$. We generate a sample of 50 000 correlated Jacobi matrices and obtain a perfect agreement between our analytic expression and the numerical simulation.

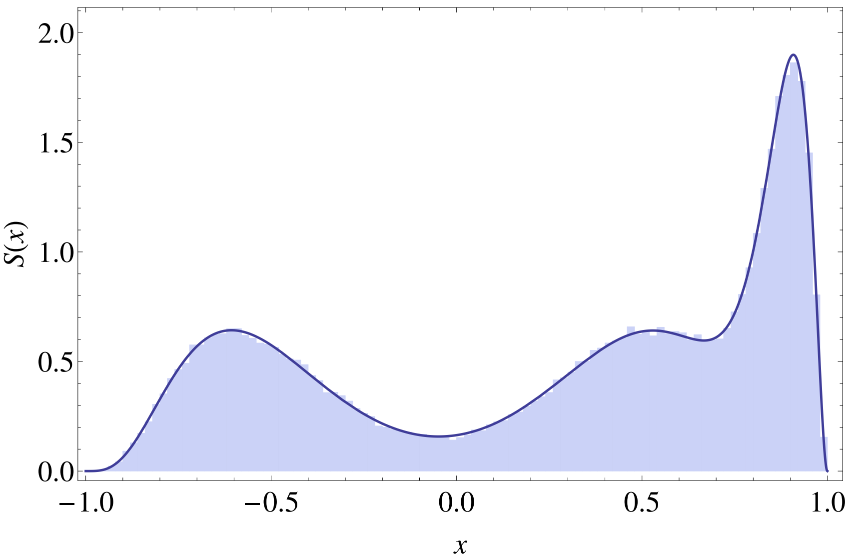


Figure 4.6: Comparison of the analytic closed-form expression (line) with the numerical simulation (histogram) of the density in the complex correlated Jacobi ensemble, where $p = 3, n_1 = 5, n_2 = 7$ and $\Lambda = \text{diag}(1/3, 2, 4.5)$.

4.3.4 Eigenvalue Density in the Real Ensemble

In the case of the real correlated Jacobi ensemble the calculation of the level density is more involved, because the dimension of the supermatrix model is $(2|2) \times (2|2)$ and square root singularities arise. Applying the generalized Hubbard Stratonovich transformation to Eq. (4.95) leads to

$$
\begin{aligned}
Z_p^{1/1}(\kappa) = K \int \mathrm{d}[\sigma]\ \mathrm{sdet}^{-(n-p)/2}\left(\sigma + \mathbf{1}_{2|2}\right) I_{n_1}(\sigma) \\
\times\, \mathrm{sdet}^{-1/2}\left(\Lambda_{\mathrm{eff}}^{-1} \otimes (\mathbf{1}_{2|2} - \hat{\kappa}) - \mathbf{1}_p \otimes (\mathbf{1}_{2|2} + \hat{\kappa})\sigma\right)\ ,
\end{aligned}
\tag{4.110}
$$

where $I_{n_1}(\sigma)$ is the supersymmetric Ingham-Siegel integral introduced in Eq. (2.97), $\hat{\kappa} = \mathrm{diag}(x_1, x_1, x \pm \imath\varepsilon, x \pm \imath\varepsilon)$ and

$$
\sigma = \begin{bmatrix} \sigma_1 & \zeta \\ -\zeta^\dagger & \sigma_2 \mathbf{1}_2 \end{bmatrix}\ ,\quad \zeta = \begin{bmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{bmatrix}\ ,\quad \sigma_1 = \begin{bmatrix} a & b \\ b & c \end{bmatrix}\ . \tag{4.111}
$$

In this parametrization of σ a, b, c, σ_2 are real commuting whereas α, β are complex anti-commuting variables. To determine the normalization constant we use that $Z_p^{1/1}(\kappa \to 0) \to 1$ and fix the measure in Eq. (4.110) to be

$$
\mathrm{d}[\sigma]\ = \mathrm{d}a\mathrm{d}b\mathrm{d}c\mathrm{d}\sigma_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^*\ . \tag{4.112}
$$

Analogously, we specify the measure in $I_{n_1}(\sigma)$. In the $\kappa \to 0$ limit the integrand in Eq. (4.110) becomes invariant, which means the rotation $\sigma \mapsto u\sigma u^\dagger$, where $u \in$ UOSp(2|2), does not change the integrand. Invariant supermatrix integrals of this kind are covered by integral theorems [152], yielding in the present case

$$
K = \frac{1}{4}\ . \tag{4.113}
$$

We insert Eq. (4.113) into the generating function (4.110) and use that the level density of the real correlated Jacobi ensemble relates to it by

$$
S(x) = \frac{1}{2\pi\imath p} \lim_{\varepsilon \to 0^+} \frac{\partial}{\partial x_1}\bigg|_{x_1 = x} \left(Z_p^{1/1}(x_1, x + \imath\varepsilon) - Z_p^{1/1}(x_1, x - \imath\varepsilon)\right)\ . \tag{4.114}
$$

The density is normalized to unity as in the previous section. Before taking the derivative in Eq. (4.114) with respect ot x_1 and doing the $\varepsilon \to 0$ limit, we expand the generating function (4.110) in terms of Grassmann variables and integrate them out.

Since the boson-boson block as well as the fermion-fermion block of $\hat{\kappa}$ are proportional to the identity matrix $\mathbf{1}_2$, the integrand is not affected by rotations of the form $\sigma_1 \mapsto \tilde{O}\sigma_1\tilde{O}^T$ with $\tilde{O} \in$ O(2). Thus, we diagonalize the boson-boson block. Accordingly, we write $\sigma_1 = OrO^T$, where $r = \mathrm{diag}(r_1, r_2)$ and $O \in$ O(2). This change of coordinates induces a decomposition of the volume form

$$
\mathrm{d}[\sigma_2]\ = |r_1 - r_2|\, \mathrm{d}[r]\ \mathrm{d}\mu(O)\ . \tag{4.115}
$$

Because of the structure of the integrand (4.110), the integral over $O(2)$ is trivial and leads to a factor of $\pi/2$. Carrying out this coordinate change in the generating function (4.110) is equivalent to replacing σ_1 by r, inserting $|r_1 - r_2|$ from Eq. (4.115) into it and multiplying the whole expression by $\pi/2$.

In the same manner, in which we did the Berezinian integrals in the complex case, we perform them here. We expand the superdeterminant of $\sigma+\mathbf{1}_{2|2}$ in the Grassmann variables α,α,β and β^* and find

$$\begin{aligned}\mathrm{sdet}^{-(n-p)/2}\left(\sigma+\mathbf{1}_{2|2}\right)=&\frac{(1+\imath\sigma_2)^{n-p}}{\mathrm{det}^{(n-p)/2}(\mathbf{1}_2+r)}\left(1+\frac{(n-p)(1+r_1)\alpha^*\alpha}{(1+\imath\sigma_2)\mathrm{det}(1+r)}\right.\\&\left.+\frac{(n-p)(1+r_2)\beta^*\beta}{(1+\imath\sigma_2)\mathrm{det}(1+r)}+\frac{\alpha^*\alpha\beta^*\beta\left((n-p)^2-(n-p)\right)}{(1+\imath\sigma_2)^2\mathrm{det}(1+r)}\right)\ .\end{aligned}\tag{4.116}$$

To expand the superdeterminant depending on the eigenvalues Λ_k, we write the tensor product as a p-fold product of superdeterminants depending on a single eigenvalue,

$$\begin{aligned}&\mathrm{sdet}^{-1/2}\left(\hat{\Lambda}_{\mathrm{eff}}^{-1}\otimes(\mathbf{1}_{2|2}-\hat{\kappa})-\mathbf{1}_p\otimes(\mathbf{1}_{2|2}+\hat{\kappa})\sigma\right)\\=&\prod_{i=1}^{p}\mathrm{sdet}^{-1/2}\left((\mathbf{1}_{2|2}-\hat{\kappa})/\Lambda_i-(\mathbf{1}_{2|2}+\hat{\kappa})\sigma\right)\ .\end{aligned}\tag{4.117}$$

For each of the factors within the product, we do a similar expansion as in Eq. (4.116) and obtain

$$\begin{aligned}&\mathrm{sdet}^{-1/2}\left((\mathbf{1}_{2|2}-\hat{\kappa})/\Lambda_i-(\mathbf{1}_{2|2}+\hat{\kappa})\sigma\right)=\frac{(1-x_1)/\Lambda_i-(1+x_1)\imath\sigma_2}{\det^{1/2}\left((1-x_0)/\Lambda_i\mathbf{1}_2-(1+x_0)r\right)}\\&\times\left(1+\frac{(1+x_0)(1+x_1)\left((1-x_0)/\Lambda_i\mathbf{1}_2-(1+x_0)r_1\right)\ \beta^*\beta}{((1-x_1)/\Lambda_i-(1+x_1)\imath\sigma_2)\det\left((1-x_0)/\Lambda_i\mathbf{1}_2-(1+x_0)r\right)}\right.\\&\left.+\frac{(1+x_0)(1+x_1)\left((1-x_0)/\Lambda_i\mathbf{1}_2-(1+x_0)r_2\right)\ \alpha^*\alpha}{((1-x_1)/\Lambda_i-(1+x_1)\imath\sigma_2)\det\left((1-x_0)/\Lambda_i\mathbf{1}_2-(1+x_0)r\right)}\right)\ ,\end{aligned}\tag{4.118}$$

where $x_0=x+\imath\varepsilon$. Finally, the expansion of the supersymmetric Ingham-Siegel integral in terms of Grassmannians was done in Ref. [163] and is given by

$$\begin{aligned}I_{n_1}(r)=&\frac{\imath^{n_1-2}4\pi}{(n_1-1)!}\Theta(r)\mathrm{det}^{(n_1-1)/2}r\left(1-\imath\left(\frac{\alpha^*\alpha}{r_2}+\frac{\beta^*\beta}{r_1}\right)\frac{\partial}{\partial\sigma_2}\right.\\&\left.-\frac{\alpha^*\alpha\beta^*\beta}{r_1r_2}\frac{\partial^2}{\partial\sigma_2^2}\right)\delta(\sigma_2)\exp(-\imath2\sigma_2)\ .\end{aligned}\tag{4.119}$$

We substitute the expansion of the superdeterminants (4.117) and (4.118) as well as the supersymmetric Ingham-Siegel integral (4.119) into the generating function (4.110). Because of the anti-commuting variables many terms drop out owing to powers of $\alpha,\alpha^*,\beta,\beta^*$ higher than one. Performing the integrals over the Grassmann variables leads to a threefold integral representing the generating function (4.110),

$$Z_p^{1/1}(\kappa)=\frac{\imath^{n_1-2}}{8(n_1-1)!}\int\limits_0^\infty\mathrm{d}r_1\int\limits_0^\infty\mathrm{d}r_2\int\limits_{-\infty}^\infty\mathrm{d}\sigma_2\frac{|r_1-r_2|\,(1+\imath\sigma_2)^{n-p}}{\mathrm{det}^{(n-p)/2}(\mathbf{1}_2+r)}$$

$$
\begin{aligned}
&\times \operatorname{det}^{(n_1-1)/2} r \prod_{i=1}^{p} \frac{(1-x_1)/\Lambda_i - (1+x_1)\imath\sigma_2}{\det^{1/2}\left((1-x_0)/\Lambda_i \mathbf{1}_2 - (1+x_0)r\right)} \left[-\frac{1}{r_1 r_2}\frac{\partial}{\partial\sigma_2} \right. \\
&+ \frac{\left((n-p)^2-(n-p)\right)}{(1+\imath\sigma_2)^2 \operatorname{det}(1+r)} - \frac{\imath(n-p)\left(r_1+r_2+2r_1r_2\right)}{r_1r_2(1+\imath\sigma_2)\operatorname{det}(1+r)}\frac{\partial}{\partial\sigma_2} \\
&- \sum_{j=1}^{p} \frac{\imath(1+x_0)(1+x_1)2r_1}{\left((1-x_1)/\Lambda_j-(1+x_1)\imath\sigma_2\right)\left((1-x_0)/\Lambda_j-(1+x_0)r_1\right)}\frac{\partial}{\partial\sigma_2} \\
&- \sum_{j=1}^{p} \frac{(n-p)(1+x_0)(1+x_1)2(1+r_2)}{\left((1-x_1)/\Lambda_j-(1+x_1)\imath\sigma_2\right)\left((1-x_0)/\Lambda_j-(1+x_0)r_2\right)} \\
&\times \frac{1}{(1+\imath\sigma_2)\operatorname{det}(1+r)} - \sum_{\substack{j,l=1\\ j\neq l}}^{p} \frac{(1+x_0)^2(1+x_1)^2}{\prod\limits_{k=j,l}\left((1-x_1)/\Lambda_k-(1+x_1)\imath\sigma_2\right)} \\
&\left.\times \frac{1}{\left((1-x_0)/\Lambda_j-(1+x_0)r_1\right)\left((1-x_0)/\Lambda_l-(1+x_0)r_2\right)} \right] \\
&\times \frac{\partial^{n_1-2}}{\partial\sigma_2^{n_1-2}}\delta(\sigma_2)\ .
\end{aligned}
\tag{4.120}
$$

The integrand (4.120) appears to be rather complex. However, we are able to do at least one more integral, the σ_2 integral. By virtue of the δ-function performing it reduces to basic calculus. We use partial integration to map the derivatives of the δ-function to the remaining integrand. Because of the δ-function, all boundary terms thereby arising vanish. To simplify the computation of the derivatives, we use the identity (4.104) and find

$$
\begin{aligned}
\frac{\partial Z_p^{1/1}(x,x_0)}{\partial x} &= \frac{(n-p)!}{16}\int_0^\infty \mathrm{d}r_1\mathrm{d}r_2 \frac{|r_1-r_2|\operatorname{det}^{(n_1-1)/2} r \operatorname{det}^{(p-n)/2}(\mathbf{1}_2+r)}{\prod_{k=1}^{p}\det^{1/2}\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1}\mathbf{1}_2-r\right)(1+x)^p} \\
&\left[2\sum_{i=0}^{p}\frac{e_{p-i}(\Lambda^{-1})(-1)^i(1+x)^{i-1}(1-x)^{p-i-1}(2i-p-px)}{(n_1-i)!(n_2-p+i)!} \right. \\
&\quad\times\left(\frac{n_1(n_1-1)}{r_1r_2}+\frac{(n_1-i)(n_1-1-i)}{(1+r_2)(1+r_1)}-\frac{2(n_1-1)(n_1-i)}{r_2(1+r_1)}\right) \\
&+4\sum_{i=0}^{p-1}\sum_{j=1}^{p}\frac{e_{p-i-1}((\Lambda^{\hat{j}})^{-1})(-1)^i(1+x)^i(1-x)^{p-i-2}}{(n_1-i-1)!(n_2-p+i+1)!\left(\frac{1-x_0}{1+x_0}\Lambda_j^{-1}-r_2\right)} \\
&\quad\times(2i+2-p-px)\left(\frac{n_1-1-i}{1+r_1}-\frac{n_1-1}{r_1}\right) \\
&+2\sum_{i=0}^{p-2}\sum_{\substack{l,j=1\\ l\neq j}}^{p}\frac{e_{p-i-2}((\Lambda^{\hat{j},\hat{l}})^{-1})(-1)^i(1+x)^{i+1}(1-x)^{p-i-3}}{(n_1-i-2)!(n_2-p+i+2)!} \\
&\quad\left.\times\frac{(2i+4-p-px)}{\left(\frac{1-x_0}{1+x_0}\Lambda_j^{-1}-r_2\right)\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1}-r_1\right)}\right]\ ,
\end{aligned}
\tag{4.121}
$$

where we take the derivative of $Z_p^{1/1}$ with respect to x_1 at x, see Eq. (4.114). We introduce $\Lambda^{\hat{j}}$ and $\Lambda^{\hat{j},\hat{l}}$ as the $p-1$ and $p-2$ dimensional matrices, obtained by removing the jth and the lth and jth entry from Λ, respectively.

The overall shape of Eq. (4.121) is similar to that obtained for the ordinary and the doubly correlated Wishart model computed in Refs. [8,163,164]. It separates into a part that has only square root singularities in r_1 and r_2, a part including in addition one $3/2$ singularity in r_2 and a third featuring two $3/2$ singularities in both variables r_1 and r_2. These are the simple, the twofold and the threefold sum in Eq. (4.121), respectively. In the analysis of Refs. [163,164] the authors used partial integration to perform the $\varepsilon \to 0$ limit. Unfortunately, the evaluation of the numerical integration of the expression obtained is rather slow. A faster evaluation is achieved by regularizing the singularities. We put a detailed discussion of this regularization and the $\varepsilon \to 0$ limit into appendix B.8. There we show that the form of the level density within the real correlated Wishart model is given by

$$S(x) = \sum_{\substack{0\leq l_1,l_2\leq p\\ l_1+l_2\in 2\mathbb{N}+1}} \int_{\mathrm{V}_{l_1}\times \mathrm{V}_{l_2}} \mathrm{d}r_1\mathrm{d}r_2 \frac{(-1)^{(l_1+l_2-1)/2} g_1(r_1,r_2)}{\prod\limits_{k=1}^{p}\sqrt{\left|\frac{1-x}{1+x}\Lambda_k^{-1}-r_1\right|\left|\frac{1-x}{1+x}\Lambda_k^{-1}-r_2\right|}}$$
$$\left(1+\sum_{l=1}^{p}\frac{f_{2,l}(r_1,r_1)}{\left(\frac{1-x}{1+x}\Lambda_l^{-1}-r_1\right)}+\sum_{\substack{i,l=1\\ i\neq l}}^{p}\frac{f_{3,l,i}(r_1,r_1)}{\left(\frac{1-x}{1+x}\Lambda_l^{-1}-r_1\right)\left(\frac{1-x}{1+x}\Lambda_i^{-1}-r_2\right)}\right), \tag{4.122}$$

where $g_1(r_1,r_2)$, $f_{2,l}(r_1,r_1)$ and $f_{3,l,i}(r_1,r_1)$ are readily determined from Eqs. (4.121), (B.60) and (B.63) .

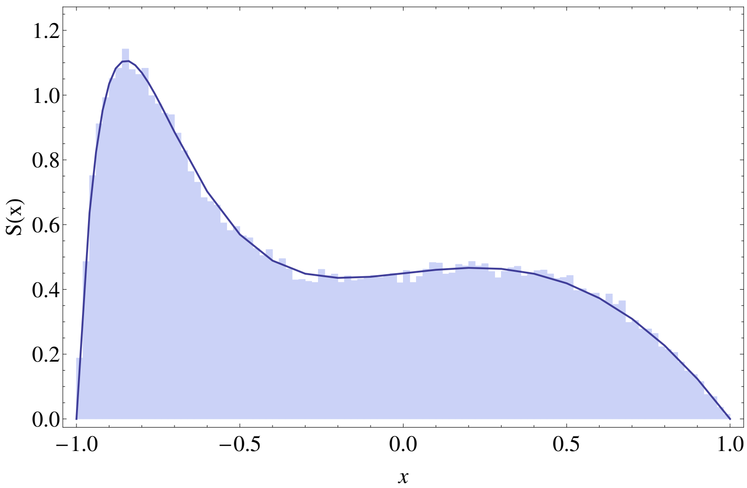


Figure 4.7: Analytic result (line) compared to numerical simulation (histogram) of the level density within the real correlated Jacobi model. For the comparision we take $p=2$, $n_1=n_2=5$ and $\Lambda_{\text{eff}}=\text{diag}(1,4)$.

For illustrating purpose and to confirm the correctness of our analytic formula for the density of the real correlated Jacobi model (4.122), we implement it into MATHEMATICA [137]. We generate an sample of 50 000 real correlated Jacobi matrices with $p = 2$, $n_1 = n_2 = 5$ and empirical eigenvalues $\Lambda_{\text{eff}} = \text{diag}(1, 4)$. The comparison is illustrated in Fig. 4.7 and shows a perfect agreement.

4.4 Asymptotic Relation Between Degenerate Real Wishart Models

In this section we study the eigenvalue statistics in the real correlated Wishart model. Therefore, we analyze the k-point correlation function which is the joint probability distribution function of k out of p eigenvalues of a Wishart matrix using the method of supersymmetry. The analysis is done under the assumption that n and p are large, while $\gamma^2 = p/n$ is fixed.

In Refs. [163, 164] the authors computed the distribution of the eigenvalues in the real correlated Wishart model. They obtained an expression in terms of a twofold integral that has for each empirical eigenvalue Λ_k a non-trivial square root singularity. This means a branch cut making the analytical computation of these integrals impossible. Similar to Eq. (4.122) in the previous section, one is left with a twofold integral representation for the density. However, the singularities become ordinary poles if one assumes a twofold degeneracy in the empirical eigenvalue spectrum. Mathematically this means that $p = 2L$ is even and the eigenvalue matrix of C is, in an appropriate basis, given by $\mathbf{1}_2 \otimes \text{diag}(\Lambda_1, \ldots, \Lambda_p)$. Physically this means that the matrix C has an involution type of symmetry, *i.e.* it commutes with a matrix that has an L-fold degenerate eigenvalue 1 and an L-fold degenerate eigenvalue -1. This is seldom justified empirically.

Nevertheless, we will show that the eigenvalue statistics in the bulk of a real correlated Wishart model consisting of $pl \times ln$ dimensional data matrices and an empirical eigenvalue matrix providing an l-fold degeneracy, approaches those of a real correlated Wishart model consisting of data matrices of size $p \times n$ without degeneracy, provided that n, p are large, p/n is fixed and the p distinct empirical eigenvalues Λ_k are the same. Hence, as long as the distinct empirical eigenvalues are the same, the bulk statistics do not depend on the degree l of degeneracy. Importantly, this implies that we can analyze statistical quantities depending on the eigenvalues of a Wishart matrix, in a model that is analytically feasible.

A few aspects we derive in the upcoming sections overlap with observations in earlier works on aspects of the spectral statistics of asymptotically large correlation matrices. In Refs. [157–159] the authors observed and emphasized the role of Eq. (4.142) for further analysis of the eigenvalue statistics. These arguments were later employed in Ref. [160] to derive the centering and rescaling for the limiting distribution of the largest eigenvalue.

The analysis done here is an extensive elaboration of the results summarized in Ref. [6]. Nevertheless, the universality of the local fluctuation of the unfolded eigenvalues we obtained thereby is not yet published and is work in progress.

In section 4.4.1, we study the k-point correlation function using the method of supersymmetry and show that it does not depend on the degree of degeneracy in the empirical eigenvalue spectrum if the matrix sizes are properly adjusted. We confirm our findings in section 4.4.2 with numerical simulations comparing eigenvalue statistics for matrix ensembles with no degeneracy and a twofold degeneracy.

4.4.1 Saddle Point Approximation

In this section, we construct a dual supermatrix model for the k-point correlation function and analyze its large n,p behavior using a saddle point approximation. To begin with, we exploit Eq. (2.35) for the real correlated Wishart model consisting of $pl \times nl$ dimensional data matrices, *i.e.* we set $H = WW^\dagger$ such that the k-point function is given by

$$R_k(x,\xi) = \frac{(lp-k)!}{(4\pi\imath)^k(lp)!} \sum_{L\in\{\pm1\}^k} \prod_{i=1}^{k} L_i \frac{\partial}{\partial j_i} \lim_{\varepsilon\to 0} Z_p^{k/k}(\kappa)\Bigg|_{j=0}, \tag{4.123}$$

where we introduce $x = \text{diag}(x_1,\ldots,x_k)$, $\xi = \text{diag}(\xi_1,\ldots,\xi_k)$, $j = \text{diag}(j_1, \ldots, j_k)$, $L = \text{diag}(L_1,\ldots,L_k)$ and $\kappa = \text{diag}(\kappa_{11},\ldots,\kappa_{k1},\kappa_{12},\ldots,\kappa_{k2})$ with $\kappa_{b1} = x_b + j_b + \xi_b/p + \imath\varepsilon L_b$ and $\kappa_{b2} = x_b - j_b + \xi_b/p + \imath\varepsilon L_b$. The generating function as shown in Eq. (4.123), can be read off from Eq. (2.35) to be

$$Z_p^{k/k}(\kappa) = K\int \text{d}[W]\ P(W|\Lambda/nl)\ \prod_{b=1}^{k} \frac{\det\left(WW^T - \kappa_{b,2}\mathbf{1}_p\right)}{\det\left(WW^T - \kappa_{b,1}\mathbf{1}_p\right)}, \tag{4.124}$$

where W is a $lp\times ln$ rectangular matrix with real entries. As introduced in section 2.3.1 the parameters L_a and j_a mimic imaginary parts and used as source variables, respectively. The normalization constant K in Eq. (4.124) is determined by the condition that $Z_p^{k/k}(j\to 0)\to 1$ such that Eq. (4.123) is correctly normalized.

We choose the parameters of the R_k such that it measures by ξ_a/p the *local fluctuations* of k out of p unfolded eigenvalues over the points x_a in the *global* spectrum. However, here unfolded does not mean, that the *level* density $R_1(x)$ is constant. It still depends on the empirical eigenvalues, because we take into account systems specific properties of the statistics.

For instance, when we set $x_1 = \cdots = x_k = x$ the k-point function measure the local fluctuation of k eigenvalues over a point x in the global spectrum. For $x_1 = \cdots = x_{m_1} \neq x_{m_1+1} = \cdots = x_k$ it measures the joined fluctuation of m_1 eigenvalues over x_1 and $k-m_1$ eigenvalues over x_k and so on and so forth.

To construct a supermatrix model dual to Eq. (4.123), we map the generating function (4.124) to superspace. Therefore, we adapt the analysis of section 2.5 and find

$$\begin{aligned} Z_p^{k/k}(\kappa) = K\int \text{d}[\sigma]\ \ \text{sdet}^{(ln-1)/2}\sigma \exp\left(-\frac{nl}{2}\text{str}\widetilde{L}\sigma\right) \\ \times\,\text{sdet}^{-l/2}\left(\mathbf{1}_p\otimes\hat{\kappa} - \Lambda\otimes\widetilde{L}\sigma\right), \end{aligned} \tag{4.125}$$

where σ is a $(2k|2k) \times (2k|2k)$-dimensional supermatrix, $\hat{\kappa} = \mathrm{diag}(\mathbf{1}_2 \otimes \kappa_1, \mathbf{1}_2 \otimes \kappa_2)$ and we introduce $\widetilde{B} = \mathbf{1}_{1|1} \otimes \mathbf{1}_2 \otimes B$ for any ordinary $k \times k$ matrix B. The supermatrix $\widetilde{L}\sigma$ has a positive definite real symmetric matrix in the boson-boson block and a fermion-fermion block lying in the circular symplectic ensemble $\mathrm{U}(2k)/\mathrm{USp}(2k)$. Employing the condition $Z_p^{k/k}(j \to 0) \to 1$ to Eq. (4.125) determines the normalization constant K to be given by

$$\begin{aligned} K^{-1} = \int \mathrm{d}[\sigma]\ \mathrm{sdet}^{(nl-1)/2}\sigma \exp\left(-\frac{nl}{2}\mathrm{str}\widetilde{L}\sigma\right) \\ \times\, \mathrm{sdet}^{-l/2}\left(\mathbf{1}_p \otimes (\widetilde{x} + \widetilde{\xi}/p + \imath\varepsilon\widetilde{L}) - \Lambda \otimes \widetilde{L}\sigma\right)\ . \end{aligned} \tag{4.126}$$

If we take the degeneracy l to be even it follows from the structure of the integrand (4.125) that all square roots singularities become ordinary poles.

Before we study the R_k for general k and, we consider $k = 1$ and analyze the density $S(x) = R_1(x)$ for large n, p but $p/n = \gamma^2$ of order $\mathcal{O}(1)$. To this end, we substitute $\hat{\kappa} = \mathrm{diag}(x + j + \imath L\varepsilon, x - j + \imath L\varepsilon) \otimes \mathbf{1}_2$ into Eq. (4.125). As described in Eq. (4.123), we differentiate the generating function for $k = 1$ with respect to j, set j to zero and take the imaginary part of it such that the level density is given by

$$\begin{aligned} S(x) = -\frac{K}{8\imath\pi p}\sum_{L_1\in\{\pm1\}} L_1 \lim_{\varepsilon\to0^+}\int \mathrm{d}[\sigma]\ \ \mathrm{sdet}^{(ln-1)/2}\sigma \exp\left(-\frac{nl}{2}\mathrm{str}\sigma\right) \\ \times \prod_{i'=1}^{p}\mathrm{sdet}^{-l/2}\left(x^+\mathbf{1}_{2|2} - \Lambda_{i'}\sigma\right)\sum_{i=1}^{p}\mathrm{str}\left(x^+\mathbf{1}_{2|2} - \Lambda_i\sigma\right)^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\right)\ , \end{aligned} \tag{4.127}$$

where $x^{\pm} = x \pm \imath L_1\varepsilon$ and $\sigma^z_{1|1} = \mathrm{diag}(1, -1)$ is the third Pauli matrix acting in $(1|1)$-dimensional superspace. We analyze $S(x)$ using the method of saddle point approximation. It is most suitable, because the dimension of the supermatrix model $(2|2)\times(2|2)$ does not vary with n, p. We separate the integrand of the density (4.127) into a part that can be written as exponent of $-n$ times a "Lagrangian" and a function that does not vary like an exponent,

$$\begin{aligned} S(x) = -\frac{K}{8\imath\pi}\sum_{L_1\in\{\pm1\}} L_1 \lim_{\varepsilon\to0^+}\int \mathrm{d}[\sigma]\ \ \exp\left(-\frac{nl}{2}\mathcal{L}(\sigma)\right) \\ \times\,\frac{1}{p}\sum_{i=1}^{p}\mathrm{str}\left(x^+\mathbf{1}_{2|2} - \Lambda_i\sigma\right)^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\right)\ , \end{aligned} \tag{4.128}$$

The Lagrangian for arbitrary k is given by

$$\mathcal{L}(\sigma) = \frac{\gamma^2}{p}\sum_{i=1}^{p}\mathrm{str}\ln\left(\widetilde{x} + \imath\varepsilon\widetilde{L} + \frac{1}{p}\widetilde{\xi} - \Lambda_i\widetilde{L}\sigma\right) + \mathrm{str}\widetilde{L}\sigma - (1 - \frac{1}{nl})\mathrm{str}\ln\sigma\ . \tag{4.129}$$

If we assume that the empirical eigenvalues Λ_k are distributed such that

$$\frac{1}{p}\sum_{i=1}^{p} f(\Lambda_i) \sim \mathcal{O}(1) \tag{4.130}$$

for n, p tending to infinity while $\gamma^2 = p/n$ is fixed and $f(\Lambda_i)$ a function of $\mathcal{O}(1)$, the leading order in n of the expansion of $\mathcal{L}$ is of order $\mathcal{O}(1)$

$$\begin{aligned}\mathcal{L}(\sigma) = \frac{\gamma^2}{p}\sum_{i=1}^{p} \mathrm{str}\ln\left(\widetilde{x} + \imath\varepsilon\widetilde{L} - \Lambda_i\widetilde{L}\sigma\right) + \mathrm{str}\widetilde{L}\sigma - \mathrm{str}\ln\sigma \\ + \frac{1}{nl}\mathrm{str}\ln\sigma + \frac{\gamma^2}{p^2 l}\sum_{i=1}^{p}\mathrm{str}\frac{\widetilde{\xi}}{\tilde{x} + \imath\varepsilon\widetilde{L} - \Lambda_i\widetilde{L}\sigma} + \mathcal{O}\left(\frac{1}{p^2}\right) \ .\end{aligned} \tag{4.131}$$

In the case of the level density, ξ is zero such that the second term in the second line of Eq. (4.131) is zero as well.

Applying the method of saddle point approximation to Eq. (4.128) requires studying the stationary points of the Lagrangian (4.129). Since it is a standard technique in random matrix theory, we summarize important intermediate steps of it only and refer for the details to Refs. [65, 104, 108, 114, 165]. The stationary points of $\mathcal{L}$ are obtained if we vary the order $\mathcal{O}(1)$ part of the Lagrangian for $k = 1$ and $\widetilde{\xi} = 0$ with respect to σ, yielding

$$\delta\mathcal{L}(\sigma) = \mathrm{str}\left(\mathbf{1}_{2|2} - \sigma^{-1} - \frac{\gamma^2}{p}\sum_{i=1}^{p}\frac{\Lambda_i}{x\mathbf{1}_4 - \Lambda_i\sigma}\right)\delta\sigma = 0 \ , \tag{4.132}$$

where we set ε to zero and $\delta\sigma$ is an arbitrary but small variation of σ. Setting $\varepsilon = 0$ is valid, because ε is infinitesimally small. Since $\delta\sigma$ is arbitrary, it turns out that

$$\mathbf{1}_{2|2} - \sigma^{-1} - \frac{\gamma^2}{p}\sum_{i=1}^{p}\frac{\Lambda_i}{x\mathbf{1}_4 - \Lambda_i\sigma} = 0 \ . \tag{4.133}$$

Although Eq. (4.132) is the original saddle point equation, we will also refer to Eq. (4.133) and its scalar version (4.134) as saddle point equation. It is worthy to mention that Eq. (4.133) is independent of the degree l of degeneracy in the empirical eigenvalues. Moreover, it is invariant under the action of the group of ortho-symplectic matrices $\mathrm{UOSp}^{(+)}(2|2)$, *i.e.* $\sigma \mapsto v\sigma v^\dagger$ where $v \in \mathrm{UOSp}^{(+)}(2|2)$. Thus, if Q_0 is a solution to Eq. (4.133), so is $v' Q_0 v'^\dagger$ for all $v' \in \mathrm{UOSp}^{(+)}(2|2)$. Instead of isolated saddle points we obtain saddle point manifolds. Because of this invariance, we can assume that σ is diagonal and study Eq. (4.133) for each individual entry, leading to

$$0 = 1 - \frac{1}{q} - \frac{\gamma^2}{p}\sum_{i=1}^{p}\frac{\Lambda_i}{x - \Lambda_i q} \ , \tag{4.134}$$

where q is an eigenvalue of Q_0. It is important to discuss the simplest case of Eq. (4.134) where correlations are absent, before discussing the general case in detail. For the uncorrelated Wishart model, *i.e.* $\Lambda = \mathbf{1}_p$, the saddle point equation (4.134) becomes

$$1 - \frac{1}{q} - \gamma^2\frac{1}{x - q} = 0, \tag{4.135}$$

It is a quadratic equation in q, possessing two solutions which are either both real or both complex. Because Eq. (4.135) is a real, the complex conjugate $q_0^*(x)$ of a solution $q_0(x)$ to Eq. (4.135) solves it as well. The two solutions are

$$q_\pm(x) = \frac{1}{2}\left(x + 1 - \gamma^2 \pm \imath\sqrt{\left((\gamma+1)^2 - x\right)\left(x - (\gamma-1)^2\right)}\right) , \tag{4.136}$$

with $q_+(x)$ (q_-(x)) lying in the upper (lower) complex half plane for $0 \leq (\gamma-1)^2 \leq x \leq (\gamma+1)^2 \leq 4$. For all other values of x it is real. According to the sum over L_1 in Eq. (4.128) only the complex saddle points contribute to the density. Thus, we consider in the following only the complex solutions to Eq. (4.135) and omit the real ones.

To reach these saddle points, we deform the integration contour in Eq. (4.128) by the residue theorem. The deformation is done by adding a contour at infinity connecting the initial and the deformed contour. Since the integrand (4.128) vanishes at infinity this contour does not contribute. By the residue theorem, the sum of all three integrals is zero whenever the combined contour does not enclose a pole of the integrand. If it happens that the integrand does not have a pole in the enclosed region, the initial and the deformed contour are equivalent. On the other hand, if this is not the case we can not deform the contour to reach the saddle point.

Because of the infinitesimally imaginary increment $\imath L_1\varepsilon$ of x, the density (4.128) exhibits for $L_1 = 1$ ($L_1 = -1$) a pole in the upper (lower) complex half plane of the boson-boson integral. Hence, by deformation we reach for $L_1 = 1$ and $L_1 = -1$ only the saddle point $q_-(x)$ lying in the lower and $q_+(x)$ lying in the upper complex half plane, respectively.

The fermion-fermion integral does not have poles such that all saddle points can be reached by contour deformations. However, in Ref. [104] the authors show that only those saddle points which have the same fermionic and bosonic degrees of freedom contribute to the leading order in p. Hence, in the vicinity of the saddle point the eigenvalue matrix of σ, can be write as

$$\sigma^D = \left(\mathrm{Re}q_+(x) - \imath L_1 \mathrm{Im}q_+(x)\right)\mathbf{1}_{2|2} + \frac{\delta q}{\sqrt{nl}} , \tag{4.137}$$

where $\delta q = \mathrm{diag}(\delta s_1, \delta s_2, \delta s_3, \delta s_3)$. Using the invariance of Eq. (4.133) under the action of $\mathrm{UOSp}^{(+)}(2|2)$, we obtain the following parametrization of the matrices σ close to the stationary points of the Lagrangian (4.131)

$$\sigma = \left(\mathrm{Re}q_+(x) - \imath L_1 \mathrm{Im}q_+(x)\right)\mathbf{1}_{2|2} + \frac{\delta\sigma}{\sqrt{nl}} = Q_0 + \frac{\delta\sigma}{\sqrt{nl}} , \tag{4.138}$$

where $\delta\sigma = u\delta q u^\dagger$ for $u \in \mathrm{UOSp}^{(+)}(2|2)$ are the so called "massive modes" and Q_0 are the "Goldstone modes" of the saddle point manifold. To compute the integral over the massive modes, we expand the Lagrangian (4.131) in the vicinity of the saddle point using Eq. (4.138) and find

$$\begin{aligned}\mathcal{L}\left(Q_0+\frac{\delta\sigma}{\sqrt{nl}}\right)&=\mathcal{L}\left(Q_0\right)+\frac{1}{\sqrt{nl}}\text{str}\left(\triangledown_\sigma\delta\sigma\right)\mathcal{L}\left(\sigma\right)\big|_{\sigma=Q_0}\\&+\frac{1}{2nl}\,\text{str}^2\left(\triangledown_\sigma\delta\sigma\right)\mathcal{L}\left(\sigma\right)\big|_{\sigma=Q_0}+\mathcal{O}(n^2)\ ,\end{aligned}\tag{4.139}$$

where $\triangledown_\sigma$ is a matrix with entries $[\triangledown_\sigma]_{ij}=\partial_{\sigma_{ji}}$. According to $Q_0\sim\mathbf{1}_{2|2}$, the first term in Eq. (4.139) is zero. Because Q_0 is a stationary point of the Lagrangian, *i.e.* $\triangledown_\sigma\mathcal{L}\left(\sigma\right)\big|_{\sigma=Q_0}=0$, the second term is zero as well. Hence, the first non-vanishing term in the expansion (4.139) is quadratic in $\delta\sigma$ and of order $\mathcal{O}(n^{-1})$. This power of n cancels the prefactor of $\mathcal{L}(\sigma)$ in the exponential (4.128) and is therefore finite for large n,p. After differentiation the term quadratic in $\delta\sigma$ becomes

$$\text{str}^2\left(\triangledown_\sigma\delta\sigma\right)\mathcal{L}\left(\sigma\right)\big|_{\sigma=Q_0}=\left(\frac{1}{q^2(x)}-\frac{\gamma^2}{(x-q(x))^2}\right)\text{str}\delta\sigma^2\ ,\tag{4.140}$$

where $q(x)=\text{Re}q_+(x)-\imath L_1\text{Im}q_+(x)$. According to the coordinate change from σ to $Q+\delta\sigma/\sqrt{nl}$ in Eq. (4.128), we get a Jacobian. In Ref. [104] it is shown that for large p the asymptotic expansion of this Jacobian is unity to leading order. Thus, we are left with a Gaussian integral over $\delta\sigma$.

If we apply a similar saddle point approximation to the normalization constant (4.126), it leads to the same Gaussian integral over massive modes such that these integrals cancel each other. Substituting the expression obtained above into Eq. (4.128), we arrive at

$$\begin{aligned}S(x)&=-\frac{1}{8\imath\pi}\sum_{L_1\in\{\pm1\}}L_1\lim_{\varepsilon\to0^+}\text{str}\left(x^+\mathbf{1}_4-Q_0\right)^{-1}\begin{pmatrix}\mathbf{1}_2&0\\0&-\mathbf{1}_2\end{pmatrix}\\&=-\frac{1}{2\imath\pi}\lim_{\varepsilon\to0^+}\left(\frac{1}{x^+-q_-(x)}-\frac{1}{x^--q_+(x)}\right)\\&=\begin{cases}\dfrac{\sqrt{\left(\xi_+-x\right)\left(x-\xi_-\right)}}{2\pi\gamma^2x}&,x\in[\xi_-,\xi_+]\\0&,\text{else}\end{cases}\ ,\end{aligned}\tag{4.141}$$

where we introduce $\xi_\pm=(\gamma\pm1)^2$. The density (4.141) is the Marčenko-Pastur distribution [40, 44, 157].

Unfortunately, for the real correlated Wishart model the scalar saddle point equation is non-trivial, because solving it is equivalent to finding all roots of a polynomial of degree $p+1$. If we rescale q in Eq. (4.134) by $-x$ we obtain

$$0=-x-\frac{1}{q}+\frac{\gamma^2}{p}\sum_{i=1}^{p}\frac{\Lambda_i}{1+q\Lambda_i}=g(q)\ ,\tag{4.142}$$

which is a classical result in high dimensional inference [157–159]. Marčenko and Pastur showed that it has a unique solution in the upper half plane, which we denote by $q_0(x)$. To confirm this observation, we study the singularities of $g(q)$.

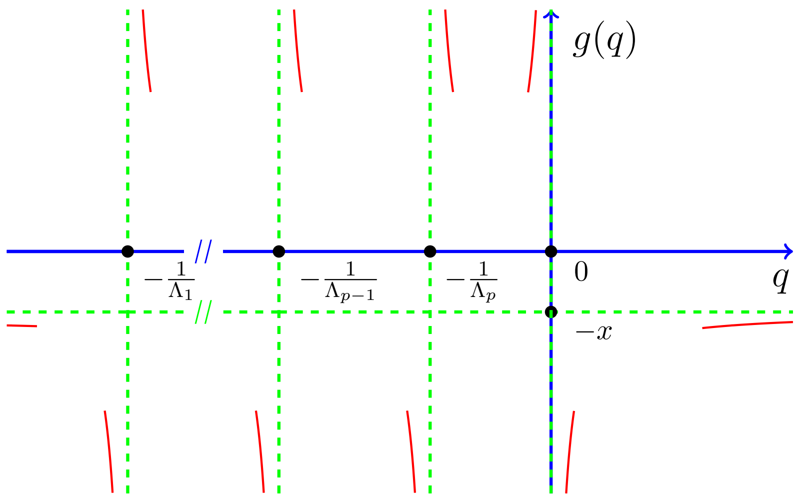


Figure 4.8: Asymptotic behavior of $g(q)$ (solid curves) at the singularities (dashed lines) and at infinity.

The polynomial $g(q)$ has $p+1$ roots and is singular at $q=-1/\Lambda_i$ for $i=1,\ldots,p$ and $q=0$. An asymptotical analysis of the behavior of $g(q)$ close to the singularities yields

$$\lim_{q\to 0^\pm} g(q) = \mp\infty\ , \quad \lim_{q\to(-1/\Lambda_i)^\pm} g(q) = \pm\infty\ , \tag{4.143}$$

where $\pm$ indicates the limit from the right, respectively, left to y. For $q\to\pm\infty$, we find that $g(q)\to -x$. We illustrate this asymptotic behavior of $g(q)$ in Fig. 4.8. Due to the asymptotic behavior, $g(p)$ crosses within each interval $(-1/\Lambda_{i+1},-1/\Lambda_i)$ the q-axis at least once for $i=1,\ldots,p-1$ and therefore has $p-1$ real roots. Since Eq. (4.134) is real, the complex conjugate $q_0^*(x)$ of a solution q_0 solves Eq. (4.134) as well. Thus, the remaining two roots are either a complex conjugate pair or both real. Because of the sum over L_1 in Eq. (4.123), only the complex solutions contribute such that we omit all real roots in the following and consider only the solution in the upper half plane denoted by $q_0(x)$.

Almost all arguments given in the uncorrelated case go through even in the case of the a correlated Wishart model (4.134). The only difference is that in the vicinity of the saddle point we linearize the saddle point manifold by

$$-\sigma/x = (\mathrm{Re}q_0(x) + \imath L_1 \mathrm{Im}q_0(x))\,\mathbf{1}_{2|2} + \frac{\delta\sigma}{\sqrt{nl}} = Q_0 + \frac{\delta\sigma}{\sqrt{nl}}\ , \tag{4.144}$$

where $\delta\sigma$ is as introduced in Eq. (4.138). We substitute this parametrization into the density (4.128) and expand the Lagrangian in $\delta\sigma$. The Gaussian integral over $\delta\sigma$ arising thereby cancels the saddle point approximation of the normalization constant. Hence, if we combine everything we arrive at

$$\begin{aligned}
S(x) &= -\frac{1}{\pi}\lim_{\varepsilon\to 0^+}\sum_{i=1}^{p}\mathrm{Im}\frac{1}{x^+ + x\Lambda_i q_0(x)} \\
&= -\frac{1}{\pi}\left\{\begin{array}{ll}\displaystyle\sum_{i=1}^{p}\mathrm{Im}\frac{1}{x + x\Lambda_i q_0(x)} & ,\mathrm{Im}q_0(x)\neq 0 \\ 0 & ,\mathrm{else}\end{array}\right. ,
\end{aligned} \tag{4.145}$$

justifying retrospectively the omission of all real roots of $g(p)$. To eliminate the sum over the empirical eigenvalues in the density (4.145), we make use of the saddle point equation (4.142). It yields

$$\sum_{i=1}^{p}\frac{1}{1+\Lambda_i q_0(x)} = -\frac{xq_0(x)+\sqrt{\xi_+\xi_-}}{\gamma^2}. \tag{4.146}$$

Hence, we replace the sum over all eigenvalues in Eq. (4.145) by the right hand side of Eq. (4.146) and take the imaginary part of it. We establish that the density is proportional to the imaginary part of the unique complex solution $q_0(x)$ in the upper complex half plane of Eq. (4.146),

$$S(x) = \frac{1}{\gamma^2\pi}\mathrm{Im}q_0(x)\ . \tag{4.147}$$

Importantly, this holds for all finite values of l and therefore confirms that the level density for large real correlated Wishart matrices is independent of the degree l of degeneracy as long as the empirical eigenvalues and the ratio of the number of rows and columns γ^2 are the same.

If we substitute the saddle point obtained for of the uncorrelated Wishart model (4.136) properly rescaled into the density (4.147), *i.e.* we set $q_0(x) = -q_-/x$, we find Eq. (4.141).

We switch to the analysis of the k-point function. Some of the arguments given for the density are valid here. We consider first the case where all points $x_1 = \cdots = x_k = y$ in the global spectrum are equal. As stated earlier, R_k measures with ξ_a the correlation between the local fluctuations of k eigenvalues located at y. We set $\kappa_{a1} = y+\xi_a/p+j_a+\imath L_a\varepsilon$ and $\kappa_{a2} = y+\xi_a/p-j_a+\imath L_a\varepsilon$ for all $a = 1,\ldots,k$ and derive the leading order contribution in p for n,p tending to infinity and $\gamma^2 = p/n$ fix of Eq. (4.123), yielding

$$\begin{aligned}
R_k(y\mathbf{1}_k,\xi) &= \frac{(-1)^k K}{(8\pi\imath)^k}\sum_{L\in\{\pm1\}^k}\prod_{i=1}^{k}L_i\int d[\sigma]\ \exp\left(-\frac{nl}{2}\mathcal{L}(\sigma)\right) \\
&\times\prod_{j=1}^{k}\frac{1}{p}\sum_{i=1}^{p}\ \mathrm{str}\frac{\mathbf{1}_{4k}}{y\mathbf{1}_{2k|2k}+\imath\widetilde{L}\varepsilon-\Lambda_i\widetilde{L}\sigma}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\otimes e^k_{jj}\right)+\ldots\ ,
\end{aligned} \tag{4.148}$$

where e^k_{lm} is a $k\times k$ matrix with zeros everywhere except in the (l,m) entry. The ... in Eq. (4.148) indicate subleading terms in p. We are integrating over the space of

$(2k|2k) \times (2k|2k)$-dimensional supermatrices σ such that $\widetilde{L}\sigma$ has a positive definite, real symmetric matrix in the boson-boson block and a fermion-fermion block in the circular symplectic ensemble $\mathrm{U}(2k)/\mathrm{USp}(2k)$, see section 2.4.2. On account of $\xi \neq 0$, the second term in the second line of Eq. (4.131) does not vanish and we find

$$\begin{aligned}\exp\left(-nl\mathcal{L}(\sigma)\right) \approx & \exp\left(-\frac{nl}{2}\mathcal{L}_{\text{eff}}(\sigma)\right) \\ & \times\ \mathrm{sdet}^{-1/2}\sigma \exp\left(-\frac{1}{2p}\sum_{i=1}^{p}\mathrm{str}\frac{\widetilde{\xi}}{\widetilde{x}+\imath\varepsilon\widetilde{L}-\Lambda_i\widetilde{L}\sigma}\right)\ .\end{aligned} \tag{4.149}$$

We introduce the effective Lagrangian $\mathcal{L}_{\text{eff}}(\sigma)$ as the order $\mathcal{O}(1)$ contribution to $\mathcal{L}$, which means the first line of the right hand side of Eq. (4.131). We use it to apply the method of saddle point approximation and vary it with respect to σ for $\varepsilon = 0$. This leads to the following equation for the stationary points of the effective Lagrangian

$$-y\mathbf{1}_{2k|2k} - Q^{-1} + \frac{\gamma^2}{p}\sum_{i=1}^{p}\frac{\Lambda_i}{\mathbf{1}_{2k|2k}+\Lambda_i Q} = 0\ , \tag{4.150}$$

where $yQ = -\tilde{L}\sigma$. It is invariant under the action of the group of pseudo-unitary-ortho-symplectic $(2k|2k) \times (2k|2k)$ matrices $\mathrm{UOSp}^{(+)}(\widetilde{L})$ given by $Q \mapsto UQU^{-1}$. Thus, if Q_0 is a solution to Eq. (4.150) so is UQ_0U^{-1}, where $U \in \mathrm{UOSp}^{(+)}(\tilde{L})$. Because of the structure of the indefinite metric,

$$\widetilde{L} = \mathbf{1}_{1|1} \otimes \mathbf{1}_2 \otimes L\ , \tag{4.151}$$

$\mathrm{UOSp}^{(+)}(\widetilde{L})$ has a subgroup $\times^k\mathrm{UOSp}^{(+)}(2|2)$, fixed by those elements of $\mathrm{UOSp}^{(+)}(\widetilde{L})$ that act on the first two factors in the tensor product above only. For a later purpose, we decompose the manifold $\mathrm{UOSp}^{(+)}(\widetilde{L})$ into the product

$$\mathrm{UOSp}^{(+)}(\widetilde{L}) = \frac{\mathrm{UOSp}^{(+)}(\widetilde{L})}{\times^k\mathrm{UOSp}^{(+)}(2|2)} \times^k \mathrm{UOSp}^{(+)}(2|2) \tag{4.152}$$

such that its elements are written as $U = TR$ for all $U \in \mathrm{UOSp}^{(+)}(\widetilde{L})$, $R \in \times^k\mathrm{UOSp}^{(+)}(2|2)$, where $T \in \mathrm{UOSp}^{(+)}(\widetilde{L})/\mathrm{UOSp}^{(+)}(2|2)$.

Hence, we first study Eq. (4.150) for diagonal Q only and then use its invariance to obtain the entire saddle point manifold. On the level of the eigenvalues of Q the analysis of Eq. (4.150), reduces to the analysis of Eq. (4.142) for each individual eigenvalue of Q. It is similar to the case of the density and we therefore apply the whole discussion presented below Eq. (4.142) to the eigenvalues of Q. Because of the sums over the L_a, only saddle points with non-vanishing imaginary parts contribute. Accordingly, the eigenvalues of the boson-boson block are fixed by the pole structure of the integrand, whereas the eigenvalues of the fermion-fermion block are either $q_0(x)$ or $q_0^*(x)$. The leading order in p of the k-point function (4.148) is due the saddle point where bosonic and fermionic eigenvalues are the same [104] such that

$$-\frac{1}{y}\tilde{L}\sigma^D = \mathrm{Re}q_0(y)\mathbf{1}_{2k|2k} + \imath\mathrm{Im}q_0(y)\tilde{L} + \frac{\delta q}{\sqrt{nl}}\ , \tag{4.153}$$

and $\delta q = \text{diag}(\delta s_{11}, \ldots, \delta s_{2k1}, \delta s_{12}\mathbf{1}_2, \ldots, \delta s_{k2}\mathbf{1}_2)$. Using the invariance of the saddle point equation (4.150), we obtain the saddle point manifold as an orbit of the action of $\text{UOSp}^{(+)}(\tilde{L})/\times^k \text{UOSp}^{(+)}(2|2)$, Ref. [165]. Thus, in the vicinity of the saddle point we find

$$-\frac{1}{y}\tilde{L}\sigma = \text{Re}q_0(y)\mathbf{1}_{2|2} + \imath\text{Im}q_0(y)T\tilde{L}T^{-1} + \frac{\delta\sigma}{\sqrt{nl}} = Q_0 + \frac{\delta\sigma}{\sqrt{nl}}\ , \tag{4.154}$$

where $\delta\sigma = (TR)\delta q(TR)^{-1}$. Expanding the effective Lagrangian with respect to $\delta\sigma$, the first and second order in this expansion are zero, because $\mathcal{L}_{\text{eff}}(Q_0) = 0$ and $\nabla_\sigma \mathcal{L}_{\text{eff}}(\sigma)|_{\sigma=Q_0} = 0$. According to $[Q_0, \delta\sigma] = 0$, the integral over the massive modes $\delta\sigma$ and the integral over the Goldstone modes Q_0 decouple such that the quadratic term in the $\mathcal{L}_{\text{eff}}(Q_0)$ expansion reduces to Eq. (4.140) with $q(x)$ replaced by $q_0(x)$. Applying the saddle point approximation to the normalization constant leads to an integral over the massive modes which cancels in leading order the $\delta\sigma$ integral of k-point function. Hence, we are left with the integral over the saddle point manifold,

$$\begin{aligned} R_k(y\mathbf{1}_k, \xi) = {} & \frac{(-1)^k}{(8\pi\imath)^k} \lim_{\varepsilon\to 0^+} \sum_{L\in\{\pm 1\}^k} \prod_{i=1}^{k} L_i \int \text{d}\mu(T) \exp\left(\frac{\text{Im}q_0(y)}{2\gamma^2}\text{str}T\widetilde{L}T^{-1}\widetilde{\xi}\right) \\ & \times \prod_{j=1}^{k} \frac{1}{p}\sum_{i=1}^{p} \text{str}\left(y\mathbf{1}_{2k|2k} + \imath\varepsilon\widetilde{L} + y\Lambda_i TLT^{-1}\right)^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\otimes e^k_{jj}\right)\ , \end{aligned} \tag{4.155}$$

where $\text{d}\mu(T)$ is the invariant Haar measure on $\text{UOSp}^{(+)}(\tilde{L})/\times^k \text{UOSp}^{(+)}(2|2)$. In the derivation of Eq. (4.155), we make use of

$$\frac{1}{p}\sum_{i=1}^{p}\frac{\widetilde{\xi}}{\mathbf{1}_{2k|2k} + Q_0} = -\frac{yQ_0 + \sqrt{\xi_+\xi_-}\mathbf{1}_{2k|2k}}{\gamma^2}\widetilde{\xi}\ , \tag{4.156}$$

which we follows from the saddle point equation (4.150) and is the matrix version of Eq. (4.146). To get rid of the remaining sum over the empirical eigenvalues Λ_i in Eq. (4.155), we employ the following observation

$$\begin{aligned} & \lim_{\varepsilon\to 0^+} \sum_{L_j\in\{\pm 1\}} L_i\left(y\mathbf{1}_{2k|2k} + \imath\varepsilon\tilde{L} + y\Lambda_i Q_0\right)^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\otimes e^k_{jj}\right) \\ & = \text{Im}\left(y\mathbf{1}_{2k|2k} + y\Lambda_i Q_0\right)^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\otimes e^k_{jj}\right) \end{aligned} \tag{4.157}$$

such that in combination with Eq. (4.156), we arrive at

$$\begin{aligned} R_k(y\mathbf{1}_k, \xi) = {} & \frac{(-1)^k\text{Im}^k q_0(y)}{(4\pi)^k \mathcal{Z}_k(\xi)} \int \text{d}\mu(T) \exp\left(\frac{\text{Im}q_0(y)}{2\gamma^2}\text{str}T\widetilde{L}T^{-1}\widetilde{\xi}\right) \\ & \times \prod_{j=1}^{k} \text{str}T\widetilde{L}T^{-1}\left(\sigma^z_{1|1}\otimes\mathbf{1}_2\otimes e^k_{jj}\right)\ . \end{aligned} \tag{4.158}$$

In Eq. (4.158), we introduce the function

$$\mathcal{Z}_k(\xi) = \int d\mu(T) \exp\left(\frac{\mathrm{Im} q_0(y)}{2\gamma^2} \mathrm{str} T\widetilde{L}T^{-1}\widetilde{\xi}\right) , \tag{4.159}$$

which is the integral over the Goldstone modes due to the saddle point approximation of the normalization constant (4.126).

The asymptotic expression of the k-point function (4.158) leads to two important observations. It is independent of the degree l of degeneracy in the empirical eigenvalue spectrum and it depends on the density (4.147) only. We rescale the unfolded eigenvalues as follows

$$\xi_a \mapsto \frac{1}{S(y)}\xi_a \quad \Leftrightarrow \quad d\xi_a \mapsto \frac{1}{S(y)} d\xi_a . \tag{4.160}$$

By this rescaling the level density of the rescaled eigenvalues ξ_a is one. The k-point correlation function obtained thereby is the same as for the real uncorrelated Wishart model. For the latter we known that it is universal such that the same is true for the real correlated Wishart model and we find

$$X_k^1(\xi_1, \ldots, \xi_k) = \lim_{\substack{p\to\infty \\ p/n=\gamma^2 \text{ fixed}}} \left(\frac{1}{S_1(y)}\right)^k R_k\left(y, \frac{\xi_1}{S_1(y)}, \ldots, \frac{\xi_k}{S_1(y)}\right) , \tag{4.161}$$

where X_k^1 is the unfolded k-point function [40,42,44]. Hence, we can study the local eigenvalue fluctuations in the bulk of the spectrum of a real correlated Wishart matrix using the universal expression X_k^1 found for the real Laguerre ensemble [166–168].

The most general case, when the k-point function is analyzed at different points x_a, can be derived from the discussion of the density and the k-point function with identical points. If we use the effective Lagrangian (4.149), the most general form of the k-point function is given by

$$\begin{aligned} R_k(x, \xi) = {} & \frac{(-1)^k K}{(8\pi\imath)^k} \sum_{L\in\{\pm 1\}^k} \prod_{i=1}^k L_i \int d[\sigma] \ \exp\left(-\frac{nl}{2}\mathcal{L}_{\text{eff}}(\sigma)\right) \\ & \times \ \mathrm{sdet}^{-1/2}\sigma \exp\left(-\frac{1}{2p}\sum_{i=1}^p \mathrm{str}\frac{\widetilde{\xi}}{\widetilde{x} + \imath\varepsilon\widetilde{L} - \Lambda_i\widetilde{L}\sigma}\right) \\ & \times \prod_{j=1}^k \frac{1}{p}\sum_{i=1}^p \ \mathrm{str}\frac{\mathbf{1}_{2k|2k}}{\widetilde{x} + \imath\widetilde{L}\varepsilon - \Lambda_i\widetilde{L}\sigma}\left(\sigma_{1|1}^z \otimes \mathbf{1}_2 \otimes e_{jj}^k\right) + \ldots , \end{aligned} \tag{4.162}$$

where the ... indicate subleading terms in p. The structural difference to Eq. (4.148) is the occurrence of $\widetilde{x}$ which is a $(2k|2k) \times (2k|2k)$ diagonal supermatrix distinct from the identity. Thus, if we vary the effective Lagrangian (4.149) with respect to σ we obtain

$$-\widetilde{x} - Q^{-1} + \frac{\gamma^2}{p}\sum_{i=1}^p \frac{\Lambda_i}{\mathbf{1}_{2k|2k} + \Lambda_i Q} = 0 , \tag{4.163}$$

where $\widetilde{x}Q = -\widetilde{L}\sigma$. In account of $\tilde{x} \not\sim \mathbf{1}_{2k|2k}$ the saddle point Eq. (4.163) is not invariant under the action of $\mathrm{UOSp}^{(+)}(\widetilde{L})$. Taking the commutator of the left hand side of the saddle point equation (4.163) with a saddle point Q_0, we find

$$[\widetilde{x}, Q_0] = 0 \ . \tag{4.164}$$

Accordingly, in an appropriate basis $Q_0 = \mathrm{diag}(Q_0^1, \ldots, Q_0^\alpha)$ as well as Eq. (4.163) become block diagonal, where Q_0^i are supermatrices of dimension $(2m_i|2m_i) \times (2m_i|2m_i)$, m_i is the degree of degeneracy of the point x_i, *i.e.* $x_{i_1} = x_{i_2} = \cdots = x_{i_m}$, and α the number of distinct points. These supermatrices solve the block diagonal saddle equation

$$-x_i \mathbf{1}_{2m_i|2m_i} - \left(Q^i\right)^{-1} + \frac{\gamma^2}{p}\sum_{i=1}^{p} \frac{\Lambda_i}{\mathbf{1}_{2m_i|2m_i} + \Lambda_i Q^i} = 0 \ , \tag{4.165}$$

where $x_i Q^i = -\widetilde{L}^i\sigma$ and $\widetilde{L}^i$ the projection of $\widetilde{L}$ onto the ith block. If no degeneracy in the points x_i is present, we obtain k copies of Eq. (4.133) with x replaced by x_i. If degeneracy is present, Eq. (4.165) is the same as the saddle point Eq. (4.150) with $k = m_i$ and $y = x_i$. For each of these block diagonal saddle point equations we find an invariance under the action of $\mathrm{UOSp}^{(+)}(\widetilde{L}^i)$ which we separate similar to Eq. (4.152). Hence, we can apply the analysis done above to the present case and arrive at

$$\begin{aligned} R_k(x;\xi) = &\prod_{i=1}^{q} \frac{(-1)^{m_i}\mathrm{Im}^{m_i} q_0(x_i)}{(4\pi)^{m_i}\mathcal{Z}_{m_i}(\xi^i)} \int d\mu(T_i) \exp\left(\frac{\mathrm{Im} q_0(x_i)}{2\gamma^2}\mathrm{str} T_i \widetilde{L} T_i^{-1}\widetilde{\xi}^i\right) \\ &\times \prod_{j=1}^{k} \mathrm{str} T_i \widetilde{L}^i T_i^{-1} \mathsf{P}_i \left(\sigma^z_{1|1} \otimes \mathbf{1}_2 \otimes e^k_{jj}\right) \mathsf{P}_i^T = \prod_{i=1}^{q} R_{m_i}(x_i;\xi^i) \ , \end{aligned} \tag{4.166}$$

where $\widetilde{\xi}^i = \mathbf{1}_{1|1} \otimes \mathrm{diag}(\xi_{i_1}, \cdots, \xi_{i_m}) \otimes \mathbf{1}_2$ is the projection of $\widetilde{\xi}$ onto the diagonal blocks, P_i are the projectors and $d\mu(T_i)$ is the Haar measure on $\mathrm{UOSp}^{(+)}(\widetilde{L}^i)/\times^{m_i}$ $\mathrm{UOSp}^{(+)}(2|2)$. Equation (4.166) confirms our previous observation that the leading order contribution in p to the k-point function does not depend on the degree l of degeneracy in the empirical eigenvalue spectrum as long as the size of the matrix model is increased in such a way that $\gamma^2 = p/n = (lp)/(ln)$ does not change. Employing the results found for the unfolding (4.161), it turns out that only the local fluctuations between eigenvalues over the same point x_i in the spectrum are universal.

4.4.2 Numerical Simulations

In the previous section, we showed by an exact calculation that the eigenvalue statistics of a real correlated Wishart matrix consisting of $p \times n$ dimensional data matrices are approached by those of a Wishart model with $lp \times ln$ dimensional data matrices and an l-fold degeneracy in the empirical eigenvalue spectrum, as long as

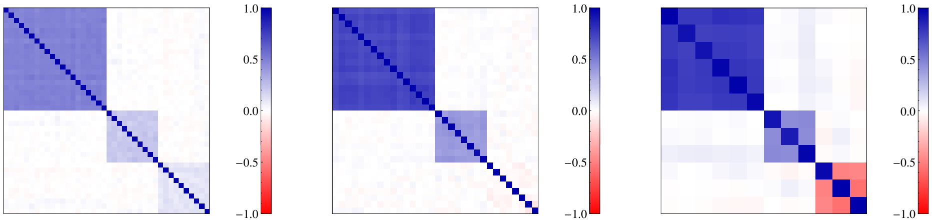


Figure 4.9: Left: Empirical correlation matrix of size 40×40 used in the first example. Middle: Empirical correlation matrix of size 32×32 used in the second example. Right: Empirical correlation matrix of size 12×12 used in the third example.

n, p are large and $p/n = \gamma^2$ is fixed. We compare numerically the level densities of both models for $l = 2$, to confirm our findings and for illustration.

To this end, we take an empirical correlation matrix as model input and generate two samples of real correlated Wishart matrices. One consisting of $p \times n$ data matrices without a degeneracy in the empirical eigenvalues and another consisting of $2p \times 2n$ data matrices with a doubly degenerate empirical eigenvalue spectrum. The distinct empirical eigenvalues Λ_k with $k = 1, \ldots, p$ are the same.

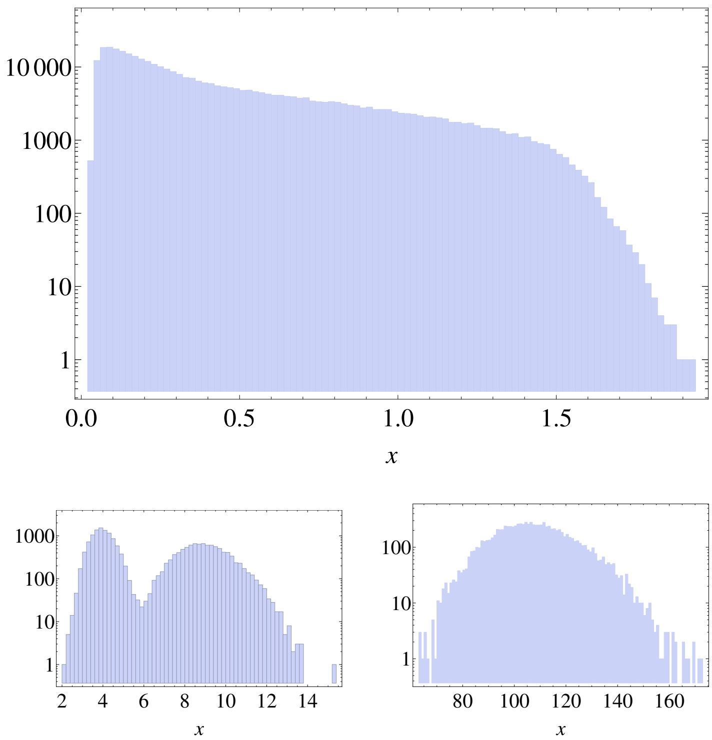


Figure 4.10: The spectrum in a real correlated 40×100 Wishart model separated into the bulk (top), the part due to second and third largest empirical eigenvalue (bottom left) and the part due to the largest empirical eigenvalue (bottom right).

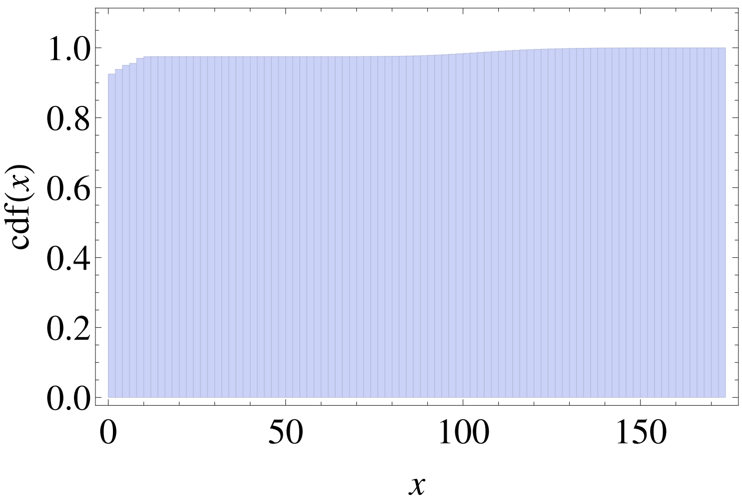


Figure 4.11: The cumulative density function corresponding to the entire density shown in Fig. 4.10.

The empirical correlation matrices, we are considering have block diagonal form. One block is of size $p/2 \times p/2$ and two are of size $p/4 \times p/4$, see Fig. 4.9 . Because of the three blocks, the empirical eigenvalue spectrum has three large eigenvalues. These will lead to a separation of the spectrum in the Wishart model into three parts. The part with the largest magnitude is the bulk of the spectrum which consists of the gross of the eigenvalues. The other two separated parts are due to the three large empirical eigenvalues. Since only a finite number of eigenvalues contributed to them they are subleading in the spectrum. In Fig. 4.10, we show a the density within a sample of $10\ 000$ real correlated Wishart matrices consisting of 40×100 dimensional data matrices. We separate the density into its three parts.

Since we compare the densities using their cumulative density functions (cdf), deviations of the latter in parts of the spectrum with less magnitude will be hidden. In Fig. 4.11 we show the cumulative density function within the sample used to generate the plots in Fig. 4.10. The leading contributions to the cdf are due to the bulk of the spectrum, because the two remaining parts of the density differ in their magnitude by orders of $\sim 10 - 10^2$. Hence, it is reasonable to compare the cdfs between the two Wishart models for each of the three parts separately. This will help to study the robustness of our approach.

For the first example we generate two samples of $10\ 000$ 40×100 and 80×200 dimensional data matrices with empirical correlation matrix shown in the left plot of Fig. 4.9. The former with non-degenerate and the latter with doubly degenerate empirical eigenvalue spectrum. In these samples, we compute the correlation matrices and analyze the level density. In Fig. 4.12, we show a comparison of the cumulative density functions in the bulk of the spectra and obtain almost perfect agreement.

Before we turn to a comparison of the cdfs in the two remaining parts, we decrease n and p to analyze the robustness of the bulk statistics. We take $p = 32$

and $n = 44$, *i.e.* $\gamma^2 = 0.53$ and generate two samples of $10\ 000$ Wishart correlation matrices. The samples consist of 32×60-dimensional data matrices without and 64×120 with doubly degeneracy in the empirical eigenvalue spectrum. The empirical correlation matrix without degeneracy used, is illustrated in the middle of Fig. 4.9. In the left plot of Fig. 4.13, we show a comparison cumulative density functions taken in the bulk of the spectra. Again, they agree almost perfectly in the entire bulk.

The third example shows the vast stability of our approach. We decrease the size of the Wishart model and take $p = 12$ and $n = 40$ so that $\gamma^2 \approx 0.3$ and compare it to a model with $p = 24$ and $n = 80$. With respect to the empirical correlation matrix illustrated in the right plot of Fig. 4.9, we generate two samples of $10\ 000$ Wishart correlation matrices and compare the cdfs within the bulk of the spectrum. Even for this ensemble of rather small correlation matrices, we observe very good agreement within the bulk of the spectrum, see the right plot in Fig. 4.13, emphasizing the overwhelming robustness of our approach.

The analytical consideration in the previous section where concerned with the bulk of the spectrum only, i.e. the part of the spectrum which contains almost all eigenvalues. Due to the three large empirical eigenvalues the spectrum of these matrices includes not only a bulk but also two seperated parts. In the top and in the bottom of Fig. 4.14 we show the comparisons of the cumulative density functions within all three examples in the second and the third parts of the spectrum, respectively. For the former only very small deviations are observed. However, for the part corresponding to the largest empirical eigenvalue the cdfs show non-negligible deviations in their tails.

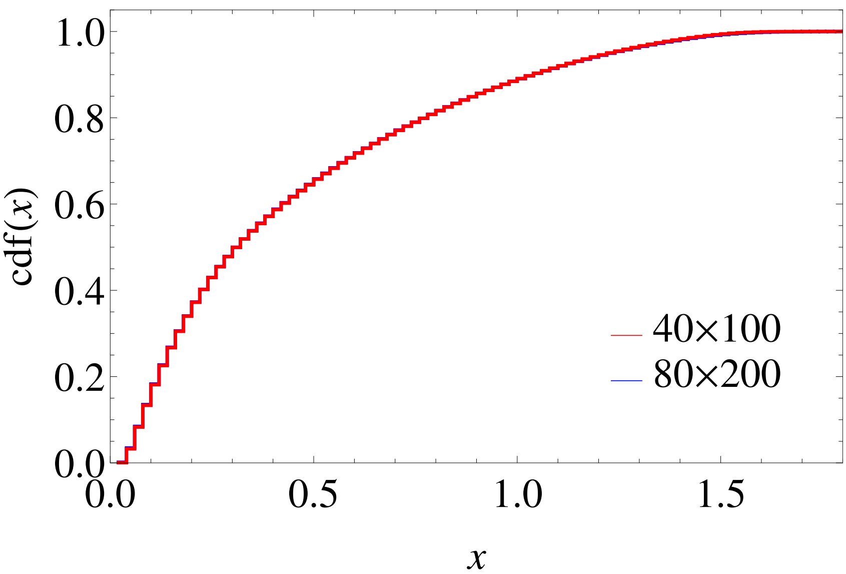


Figure 4.12: Comparison of the cumulative density function (cdf) in the bulk of a sample of $10\ 000$ 40×40 (blue histogram) and 80×80 (red histogram) Wishart matrices with $n = 100$ and $n = 200$, respectivley, *i.e.* $\gamma^2 = 0.4$.

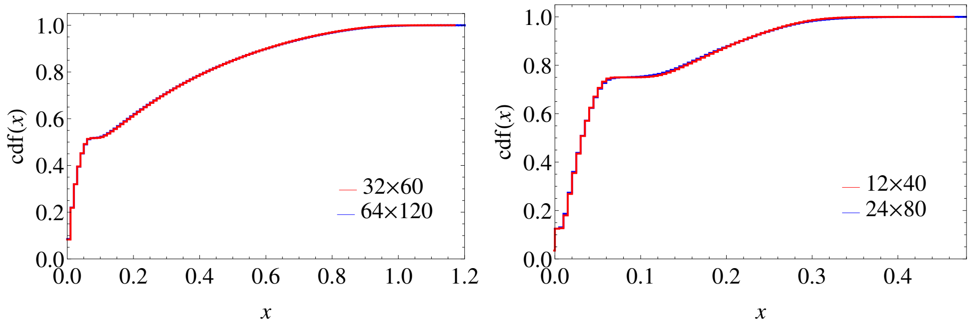


Figure 4.13: Left: Comparison of the cumulative density function (cdf) in the bulk of two samples of 10 000 Wishart matrices consisting of 32×60 (blue histogram) and 64×120 (red histogram) dimensional data matrices, *i.e.* $\gamma^2 = 0.53$. Right: Comparison of the cumulative density function (cdf) in the bulk of two samples of 10 000 Wishart matrices consisting of 12×40 (blue histogram) and 24×80 (red histogram) dimensional data matrices, *i.e.* $\gamma^2 = 0.3$.

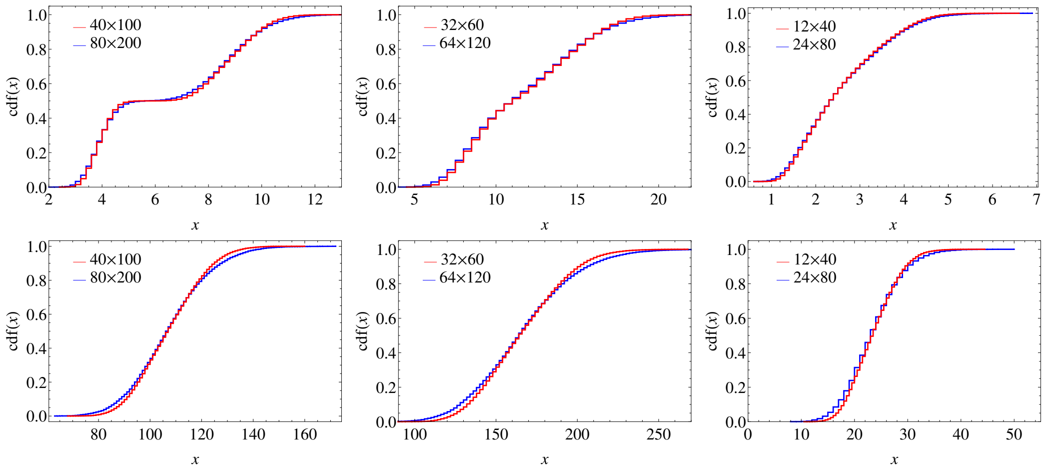


Figure 4.14: *Top*: Cumulative density function in the second part of the spectrum within the first (left), second (middle) and third (right) sample. *Bottom*: Cumulative density function in the third part of the spectrum within the first (left), second (middle) and third (right) sample.

4.5 Summary Chapter 4

In this chapter we approached the eigenvalue statistics in the correlated Wishart and Jacobi model exclusively with the method of supersymmetry. In particular, we considered the statistics of the smallest and the largest eigenvalue of a correlated Wishart matrix, the level density in the real and complex correlated Jacobi ensemble and obtained an asymptotic relation between the eigenvalue statistics in the bulk of a degenerate and a non-degenerate real correlated Wishart model.

Like any statistical quantity depending on the eigenvalues of a correlated Wishart matrix only, the quantities considered here can not be analyzed using standard techniques such as orthogonal polynomials. This is because the empirical correlation matrix destroys the invariance of the matrix model such that no closed-form expression of joint eigenvalue distribution function is known, except for the complex ensemble.

The smallest eigenvalue distribution within the real and complex correlated Wishart model was studied using a related gap probability. We constructed an average over a non-invariant Wishart ensemble dual to the eigenvalue integral representation of the gap probability. It turned out that, because of a freedom in the choice of the rectangularity in the data matrices, we observed infinitely many dual Wishart matrix models representing the same quantity. Each of which can be mapped to a supermatrix model, inducing a similar duality between supermatrix models. However, only one proved to be the best choice, because it led not to a non-invariant supermatrix model but to an ordinary, invariant matrix model. We solved it for the real and the complex ensemble and derived a closed-form expression for the gap probability and therefore for the smallest eigenvalue distribution. Importantly, we were able, for the first time, to fully circumvent the orthogonal Itzykson-Zuber integral and moreover to obtain a Pfaffian structure even though a correlation structure was present. Because we expressed the gap probability for the real and the complex ensemble as a Pfaffian and as a determinant of size $\sim \mathcal{O}(n-p)$, respectively, our expressions are imminently suitable to analyze the microscopic limit. For this limit we obtained that the statistics of the smallest eigenvalue become universal on a local scale and behave as those of the uncorrelated Wishart model. We confirmed our analytical findings with numerical simulations.

The statistics of the largest eigenvalue of a real or a complex correlated Wishart matrix is approached analogous to the smallest eigenvalue in terms of a gap probability. By similar arguments, we found infinitely many possibilities to express this gap probability in terms of a Wishart model average. We make the same choice as above and obtained two ordinary, invariant matrix ensembles. They differ in their dimensions and symmetry classes. The first model we found consists of $2N/\beta \times 2N/\beta$ dimensional real quaternion self-dual matrices for $\beta = 1$ or Hermitian matrices for $\beta = 2$, where we have to take $N \to \infty$ at the end of the calculation. It has the advantage that in the real case, because of the twofold degeneracy in the eigenvalues, square roots appearing in the integrand collapse to integer powers at the cost of an infinite dimensional matrix model. The second ensemble consists of $2(p+2-\beta)/\beta \times 2(p+2-\beta)/\beta$ dimensional real symmetric matrices for $\beta = 1$ and Hermitian matrices for $\beta = 2$. Here N is only a parameter. However, square roots appearing in the integrand are still there and can not be simplified further. For the complex case we showed that we are able to recover the known results for the gap probability which we confirmed by numerical simulations.

For the level density in the correlated Jacobi ensemble we showed that it has two dual supermatrix models. One consisting of an average over two supermatrices and another in terms of a single supermatrix average. The latter is based on the observations that the eigenvalue statistics of the Jacobi ensemble are equivalent to those of a Lorentzian model. Because the calculation of the characteristic function with respect to this distribution is highly non-trivial, we mapped the generating function to a supermatrix model using a projection formula. It yielded a Lorentzian distributed supermatrix ensemble. We solved the latter for the real and the complex case and derived a twofold integral representation and a closed-form expression for the level densities, respectively.

In the last part of this chapter, we discussed an asymptotic relation between the bulk eigenvalue statistics of two real correlated Wishart models with and without degeneracy in the empirical eigenvalues. We showed that if the ensembles are drawn from $p \times n$, respectively, $lp \times ln$ dimensional data matrices and the empirical eigenvalues are the same up to an l-fold degeneracy in the latter model, the eigenvalue statistics of the Wishart matrices to leading order in p are identical as long as $p/n = \gamma^2$ is finite. This asymptotic duality was derived by considering the k-point function using supersymmetry and a saddle point approximation. It suggests to study aspects of spectral statistics in the bulk of a $p \times n$ real correlated Wishart model within an ensemble with double degeneracy which is analytically much more feasible. Thereby, we observed that the local fluctuations of the unfolded eigenvalues on the scale of the mean level spacing are universal. We confirmed our findings with numerical simulations emphasizing the robustness of our approach. Moreover, we found this relation to hold at least approximately for parts of the spectrum which are not in the bulk.

CHAPTER 5

A New Approach to Correlated Ensembles

In the analysis of statistical quantities depending on eigenvalues of correlated Wishart matrices only, standard techniques fail to apply. The reason for this break down is the occurrence of the Itzykson-Zuber integral. As a consequence, mathematicians developed the Jack-polynomials in order to treat this integral, see Ref. [41, 100–102, 169] and references therein. Despite of the overwhelmingly rich structure of the Jack polynomials, they have the drawback that evaluation can be achieved only on the computer [103]. This is because they have to be constructed recursively.

In terms of these polynomials some aspects of the Wishart model discussed in this chapter were considered recently. The cumulative density function of the largest eigenvalue as well as the probability related to the smallest eigenvalue distribution were computed in Refs. [141, 142]. Besides these exact expressions, the limiting statistics of the former was studied in Refs. [98, 99, 160, 170, 171], where the authors proved that in some cases it is given by the Tracy-Widom distribution.

We develop and apply in this chapter a new approach to study these statistical quantities using standard methods from random matrix theory. Contrary to the known results, we will be able to uncover Pfaffian and determinantal structures in terms on known functions. These will yield insights in the dependence of these observables on the empirical eigenvalues.

Our discussion on the Fourier approach and its application to gap probabilities for the smallest and largest eigenvalue as well as the analysis of their limiting distributions is based on and extends the results of Ref. [4]. The extensive elaboration of the exact expressions for the gap probabilities in the correlated real, complex and real quaternion Wishart model for finite values of n and p presents not yet published work in progress.

The Fourier approach developed in section 5.1 is based on a Fourier transform and exploits the dyadic structure of the Wishart correlation matrix to map non-invariant Wishart models onto invariant matrix averages. We apply this in section 5.2 to a gap probability related to the largest eigenvalue distribution and find an invariant matrix model. For all three ensembles we discover previously unknown Pfaffian and determinantal expressions in terms of known functions. In section 5.3, we apply the Fourier approach to a gap probability related to the distribution of the smallest eigenvalue. We exploit the invariant matrix model obtained thereby as well as the one found for the gap probability related to the distribution of the largest eigenvalue to study in section 5.4 the limiting statistics of these extreme eigenvalues. We show that in some cases these follow the Tracy-Widom distribution. We close this chapter with a summary in section 5.5.

5.1 Fourier Approach

We present a comprehensive approach to analyze statistical quantities depending on the eigenvalues of a correlated Wishart matrix only. If these quantities possess a representation as an average over an ensemble of Wishart matrices we show under certain assumptions that they can equivalently be expressed in terms of an invariant matrix model. The ensemble we obtain thereby, consists of real symmetric, Hermitian or real quaternion self-dual matrices, depending on the symmetry class of $WW^\dagger$.

To begin with, we denote by $\mathcal{O}(WW^\dagger)$ an function of a correlated Wishart matrix. We assume that it is invariant under the action of $\mathrm{G}_p \times \mathrm{G}_n$ on $\mathrm{Mat}_{p\times n}(\mathbb{K})$, given by $W \mapsto UWV^\dagger$, where $U \in \mathrm{G}_p$, $V^\dagger \in \mathrm{G}_n$ and that it has the property $\mathcal{O}(WW^\dagger) = \mathcal{O}(W^\dagger W)$. Under these modest assumptions, we show that the average of this quantity within the correlated Wishart model which is non-invariant,

$$\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = K\int d[W]\ \mathcal{O}\left(WW^\dagger\right) P(W|\hat{\Lambda})\ , \tag{5.1}$$

can be mapped to an invariant matrix model. We make use of the positive definiteness of the correlation matrix and the invariance of the statistical quantity and replace $\mathcal{O}(WW^\dagger)$ by $\mathcal{O}(W^\dagger W)$ in Eq. (5.1). The resulting argument of the observable $W^\dagger W$ is replaced by a $n \times n$ matrix H in the same symmetry class as $W^\dagger W$ using a δ-function. After an appropriate H contour deformation, we exchange the W and the H integrals. The W integral obtained thereby is a Gaussian integral,

$$\begin{aligned}\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = K\int & d[H,Q]\ \mathcal{O}\left(Q\right)\exp\left(\imath \mathrm{tr} HQ\right)\\ \times\int & d[W]\ \exp\left(-\imath\mathrm{tr}HW^\dagger W\right) P(W|\hat{\Lambda})\ ,\end{aligned} \tag{5.2}$$

where H and Q are $n \times n$ real symmetric, Hermitian or real quaternion self-dual matrices. The distinguishing feature of Eq. (5.2) is that H couples to $W^\dagger W$ whereas Λ^{-1} to $WW^\dagger$. Accordingly, we find by using the invariance of the observable and the flat measure that the H integral is invariant with respect to the action of G_n. The W integral can be written as Gaussian integral over a np-dimensional vector with entries in $\mathbb{K}$, where the matrix defining the bilinear form is given by $\mathbf{1}_n \otimes \Lambda^{-1} + \frac{2\imath}{\beta} H \otimes \mathbf{1}_p$. If we perform it, the invariant matrix integral dual to Eq. (5.1) is given by

$$\left\langle \mathcal{O}\left(WW^\dagger\right)\right\rangle = \int d[H]\ \frac{\mathcal{F}_n\left(H\right)}{\det^{1/\gamma_1}\left(\mathbf{1}_{\gamma_2 n}\otimes\mathbf{1}_{\gamma_2 p} + \frac{2\imath}{\beta}H\otimes\Lambda\right)}\ , \tag{5.3}$$

where

$$\mathcal{F}_n\left(H\right) = \frac{1}{(2\pi)^{\mu}} \int \mathrm{d}[Q]\ \mathcal{O}\left(Q\right) \exp\left(\imath \mathrm{tr} HQ\right) \ , \tag{5.4}$$

$\mu = p(p{+}1)/2, p^2, p(2p{-}1)$ and Q is the same kind of matrix as H. The normalization constant is determined by taking $\mathcal{O}(WW^\dagger) = 1$ and employing the normalization of $P(W|C)$. We provide in the upcoming sections two examples where this approach leads to unknown expressions for gap probabilities related to the distribution of the smallest and largest eigenvalue.

5.2 Largest Eigenvalue Statistics

In section 4.2 we showed using bosonization that the gap probability corresponding to the largest eigenvalue distribution (2.36) in the real and the complex correlated Wishart ensemble can be expressed in terms of an eigenvalue integral. For the complex ensemble we were able to recover the known results. However, the limit process involved in the calculation made it impossible to observe a closed-form solution for the real correlated Wishart ensemble. We partially solve this problem in the following.

We construct the invariant matrix model in two ways. First we use a matrix integral representation for the Heaviside Θ-function of matrix-argument in section 5.2.1. In section 5.2.2, we find a similar expression using the results of section 5.1. Employing the invariance of this model it reduces to an eigenvalue integral, which we discuss for each symmetry class separately. In sections 5.2.3 and 5.2.4 we derive an exact expression for the gap probability in the complex and the real quaternion ensemble, respectively. In section 5.2.5 and 5.2.6 we discuss the real case with and with out doubly degeneracy.

5.2.1 Invariant Dual Matrix Model

We consider the gap probability (2.40) to find all eigenvalues of a correlated Wishart matrix within the interval $[0,t]$ and rescale W by $\sqrt{t}$ leading to

$$E_p^{(\beta)}([0,t];p) = K t^{np\beta/2} \int \mathrm{d}[W]\ P(W|t\hat{\Lambda}^{-1}) \Theta(\mathbf{1}_{p\gamma_2} - WW^\dagger) \ , \tag{5.5}$$

where K is an overall normalization constant yet to be determined. The Θ-function is invariant under the left-right action of $\mathrm{G}_p \times \mathrm{G}_n$ on $\mathrm{Mat}_{p\times n}(\mathbb{K})$, $W \mapsto UWV^\dagger$. Thus, we can replace $\mathbf{1}_{\gamma_2 p} - WW^\dagger$ by $\mathbf{1}_{\gamma_2 n} - W^\dagger W$ in its argument and obtain

$$E_p^{(\beta)}([0,t];p) = K t^{np\beta/2} \int \mathrm{d}[W]\ P(W|t\hat{\Lambda}^{-1}) \Theta(\mathbf{1}_{\gamma_2 n} - W^\dagger W) \ . \tag{5.6}$$

The Θ-function has a matrix integral representation [172], which is a generalization of the contour integral representation of the Heaviside Θ-funtion on the real line [136]. If we assume that A is a matrix in one of Dyson's three symmetry classes, the integral representation of the Θ-function is known as the Ingham-Siegel integral and is given by [172]

$$\Theta(A) \sim \int \mathrm{d}[H]\ \frac{\exp\left(\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right) A\right)}{\det^{\alpha/\gamma_1}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)}\ . \tag{5.7}$$

Instead of $\imath H + \mathbf{1}_{\gamma_2 n}$ we could also use $\imath H + \mu \mathbf{1}_{\gamma_2 n}$, where $\mu \in \mathbb{R}_{>0}$, because the Θ-function is independent of μ. In appendix D, we provide a very simple and previously unknown derivation of Eq. (5.7) including normalization constants. The domain of integration in Eq. (5.7) is the set of real symmetric, Hermitian or real quaternion self-dual $n \times n$ matrices and we introduce

$$\alpha = n - 1 + 2/\beta = n + \begin{cases} 1 & ,\beta = 1 \\ 0 & ,\beta = 2 \\ -1/2 & ,\beta = 4 \end{cases}\ . \tag{5.8}$$

Substituting the matrix integral (5.7) into the gap probability (5.5) and exchanging the integral measures we are left with

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = K t^{np\beta/2} \int \mathrm{d}[H]\ \frac{\exp\left(\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)\right)}{\det^{\alpha_\beta/\gamma_1}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)} \\ \times \int \mathrm{d}[W]\ \exp\left(-\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right) W^\dagger W - \frac{\beta t}{2}\mathrm{tr} W W^\dagger \Lambda^{-1}\right) \end{aligned}\ . \tag{5.9}$$

The W integral reduces to an ordinary Gaussian integral over a np-dimensional vector with entries in $\mathbb{K}$. If we perform it, we find

$$\begin{aligned} & E_p^{(\beta)}([0,t];p) = K t^{np\beta/2} \\ & \times \int \mathrm{d}[H]\ \frac{\det^{-\alpha/\gamma_1}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right) \exp\left(\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)\right)}{\det^{1/\gamma_1}\left(\frac{t\beta}{2}\mathbf{1}_{\gamma_2 n} \otimes \Lambda^{-1} + \left(\imath H + \mathbf{1}_{\gamma_2 n}\right) \otimes \mathbf{1}_p\right)}\ . \end{aligned} \tag{5.10}$$

Since Λ and H are acting in different spaces, the matrix model (5.10) is invariant under action $H \mapsto U' H (U')^\dagger$ for all $U' \in \mathrm{G}_n$. Thus, we diagonalize $H = U(\mathbf{1}_{\gamma_2} \otimes Y)U^\dagger$, where $U \in \mathrm{G}_n$ and $Y = \mathrm{diag}\,(y_1, \ldots, y_n)$ is the matrix of distinct eigenvalues, and study the eigenvalue integral. Diagonalization of H induces a decomposition of the volume form [40], given by

$$\mathrm{d}[H]\ \sim |\Delta_n(Y)|^\beta\, \mathrm{d}[Y]\ \mathrm{d}\mu(U)\ , \tag{5.11}$$

where $\mathrm{d}\mu(U)$ is the Haar measure on G_n. Because of the invariance of the integrand, the resulting group integral is trivial and only contributes to the overall normalization constant. The eigenvalue integral representation of the gap probability (2.40) arising thereby reads

$$\begin{aligned} & E_p^{(\beta)}([0,t];p) = K t^{np\beta/2} \\ & \times \int \mathrm{d}[Y]\ \frac{|\Delta_n(Y)|^\beta \det^{-\alpha\beta/2}\left(\imath Y + \mathbf{1}_n\right) \exp\left(\gamma_2 \mathrm{tr}\left(\imath Y + \mathbf{1}_n\right)\right)}{\det^{\beta/2}\left(\frac{t\beta}{2}\mathbf{1}_n \otimes \Lambda^{-1} + \left(\imath Y + \mathbf{1}_n\right) \otimes \mathbf{1}_p\right)}\ , \end{aligned} \tag{5.12}$$

where the individual integrals are over the entire real line. Because of the exponent $\beta/2$ of the determinant in the denominator in Eq. (5.12), only $\beta = 1$ and $\beta = 4$ can be studied using standard techniques.

5.2.2 Fourier Approach to $E_p^{(\beta)}([0,t];p)$

To illustrate the simplicity of the Fourier approach, see section 5.1, we apply it to the gap probability (5.5). The statistical quantity we are averaging is $\Theta\left(\mathbf{1}_{\gamma_2 p} - WW^\dagger\right)$. It is not a function but a distribution on the space of Wishart matrices and it meets all requirements. It is invariant and $\Theta\left(\mathbf{1}_{\gamma_2 p} - WW^\dagger\right) = \Theta\left(\mathbf{1}_{\gamma_2 n} - W^\dagger W\right)$. Its Fourier transform is a standard integral and can readily be done

$$
\begin{aligned}
&\mathcal{F}_n[\mathcal{O}]\left(H - \imath\mathbf{1}_n\right) \sim \int \mathrm{d}[Q]\ \Theta\left(\mathbf{1}_{\gamma_2 n} - Q\right)\exp\left(\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)Q\right)\\
&\sim \exp\left(\mathrm{tr}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)\mathbf{1}_{\gamma_2 n}\right)\ \det{}^{-(n+2/\beta-1)/\gamma_1}\left(\imath H + \mathbf{1}_{\gamma_2 n}\right)\ ,
\end{aligned}
\tag{5.13}
$$

where $n + 2/\beta - 1 = \alpha$. If we substitute the Fourier transform of our statistical quantity (5.13) into Eq. (5.3), we arrive at Eq. (5.10).

5.2.3 Complex Case

A crucial difference between the complex and the other symmetry class is that an analytic closed-form expression exists for the Harish-Chandra-Itzykson-Zuber integral [95,96]. Thus, we can compare results obtained from expressions discussed earlier with known results.

In appendix C.1, we summarize the derivation of a closed-form expression for the gap probability (2.40) using the Harish-Chandra-Itzykson-Zuber integral [95,96] given in Ref. [143]. There, we obtain that the gap probability has a determinantal structure with a simple matrix kernel,

$$
E_p^{(2)}([0,t];p) = \frac{(-1)^{p(p-1)/2}}{\Delta_p(\Lambda)}\det\left[\Lambda_j^{i-1}\left(1 - \sum_{m=0}^{n-p+i-1}\frac{\exp\left(-t/\Lambda_j\right)}{m!}\left(\frac{t}{\Lambda_j}\right)^m\right)\right]\ ,
\tag{5.14}
$$

where $i,j = 1,\ldots,p$. We derived this expression using Berezin integrals in section 4.2 and compared it to Monte-Carlo Simulations confirming its correctness, Fig. 4.5.

We derive starting from Eq. (5.12) several different and unknown representations of the gap probability and show that we can recover the known results (5.14). From the analysis of section 5.2.1, the gap probability is given by an eigenvalue integral of a $n\times n$-dimensional Hermitian matrix (5.12), which in the present case reads

$$
E_p^{(2)}([0,t];p) = Kt^{np}\int\frac{\mathrm{d}[Y]\ \Delta_n^2\left(\imath Y + \mathbf{1}_n\right)\exp\left(\mathrm{tr}(\imath Y + \mathbf{1}_n)\right)}{\det^n(\imath Y + \mathbf{1}_n)\prod_{k=1}^p\det\left(t/\Lambda_k\mathbf{1}_n + \imath Y + \mathbf{1}_n\right)}\ .
\tag{5.15}
$$

We make use of the translation invariance of the Vandermonde determinant to replace $\Delta_n\left(Y\right)$ by $\Delta_n\left(\imath Y + \mathbf{1}_n\right)$. The eigenvalue integral (5.15) can be seen as an averaged product of inverse characteristic polynomials with respect to the weight

$$
w(y) = \frac{\exp(\imath y + \mathbf{1}_n)}{(\imath y + \mathbf{1}_n)^n}\ .
\tag{5.16}
$$

These kinds of averages are standard in random matrix theory and were considered in Refs. [107,110,111,173] and references therein. We apply the ideas of Ref. [111] and cast the gap probability into an expression with determinantal structure and a kernel of size $p \times p$. The detailed calculation is done in appendix C.2 and leads to

$$E_p^{(2)}([0,t];p) = \frac{(-1)^{np-p(p+1)/2} t^{pn-p(p+1)/2}}{\det^{n-p}\Lambda \ \Delta_p(\Lambda)} \det\left[K_{ik}^{\beta=2}(\Lambda)\right] \ , \tag{5.17}$$

where $1 \le i,k \le p$ and

$$\begin{aligned} K_{i'k}^{\beta=2}(\Lambda) = {} & \left(\frac{-\Lambda_k}{t}\right)^{n-i'} \exp\left(-\frac{t}{\Lambda_k}\right) + \sum_{m=n-p}^{n-i'} \frac{(-\Lambda_k/t)^m}{(n-i'-m)!} \\ & - \sum_{f=1}^{n-p} \left(\frac{-\Lambda_k}{t}\right)^{n-p-f} \exp\left(-\frac{t}{\Lambda_k}\right) \sum_{l=0}^{f-1} \frac{(-1)^l}{(p+f-i'-l)!} \ . \end{aligned} \tag{5.18}$$

It is not obvious how we get from Eq. (5.17) to Eq. (5.14). Although they look structurally quite similar, it seems an unsurmountable task to reorder the kernel. Hence, we present an alternative way to derive Eq. (5.14) from Eq. (5.15).

The second approach is more appropriate to recover the known result. We apply the analysis done in appendix C.2 to obtain Eq. (C.8). It consists of a determinant with $n \times n$ dimensional kernel. Accordingly, we are allowed to choose a different basis such that

$$(\imath z + 1)^i \mapsto R_i(z). \tag{5.19}$$

The monic polynomials $R_i(z) = z^i + \ldots$, are of degree i and orthogonal with respect to the scalar product

$$\langle R_i | R_j \rangle = \int_{-\infty}^{\infty} \mathrm{d}x R_i(x) R_j(x) w(x) = r_i \delta_{ij} \tag{5.20}$$

and the weight (5.16), where r_j are normalization constants. The orthogonality of the polynomials yields that if we do this particular base change in Eq. (C.8), the gap probability (5.15) becomes

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{Kt^{pn}}{\Delta_p(t\Lambda^{-1})} \det\left[\int_{-\infty}^{\infty} \mathrm{d}z \frac{R_{i'-1}(z)\, w(z)}{\left(t\Lambda_k^{-1} + \imath z + 1\right)} \ \middle| \ \langle R_{i'-1} | R_{l-1} \rangle \right] \\ &= \frac{Kt^{pn}}{\Delta_p(t\Lambda^{-1})} \det\left[\int_{-\infty}^{\infty} \mathrm{d}z \frac{R_{i-1}(z)\, w(z)}{\left(t\Lambda_k^{-1} + \imath z + 1\right)} \right] \ , \end{aligned} \tag{5.21}$$

where $i = n-p+1, \ldots, n$ and $k = 1, \ldots, p$. Hence, we reduce the dimension of the determinant from $n \times n$ to $p \times p$ and are left with the calculation of the Cauchy transform of orthogonal polynomials only. In appendix C.3, we show that it is given by

$$\int_{-\infty}^{\infty} \mathrm{d}z \frac{R_{i-1}(z)\, w(z)}{\left(t\Lambda_k^{-1} + \imath z + 1\right)} \sim \int_{0}^{1} \mathrm{d}x \exp\left(-\frac{t}{\Lambda_k} x\right) x^{i-1}(1-x)^{n-i} = \varphi_{i-1}^{n-i}\left(t\Lambda_k^{-1}\right) \ , \quad (5.22)$$

where we introduce the function

$$\varphi_s^m(z) = \sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!} \left(\frac{(s+l)!}{z^{s+l+1}} - \sum_{j=0}^{s+l} \frac{(s+l)!}{j!} \frac{\exp(-z)}{z^{s+1-j}} \right) \ . \quad (5.23)$$

It has two properties that become important later. We can express each φ_s^m in terms of a sum of φ_s^0s,

$$\varphi_s^m(z) = \sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!} \varphi_{s+l}^0(z) \ . \quad (5.24)$$

Moreover, it has a simple large $z \gg 1$ expansion, which we read off from the integral representation (5.22),

$$\varphi_s^m(z) = s! \left(\frac{1}{z}\right)^{s+1} + \mathcal{O}(z^{-s+2}) \ . \quad (5.25)$$

According to the property of the determinant being linear under the multiplication of rows or columns by scalars, we can absorb the proportionality constant in Eq. (5.22) into the overall normalization constant. Hence, replacing the Cauchy transform in the determinant (5.21) by Eq. (5.22) leads to

$$E_p^{(2)}([0,t];p) = \frac{K_{p\times n} t^{pn}}{\Delta_p(t\Lambda^{-1})} \det\left[\varphi_{n-p+i-1}^{p-i}\left(t\Lambda_k^{-1}\right)\right] \ , \quad (5.26)$$

where $1 \leq i, k \leq p$. Before we determine the normalization constant, we show that the gap probability (5.26) coincides with the known result (5.14). To do so, we compare the last three rows in the determinant (5.26) using the expansion (5.24) and find

$$i = p: \ \ \varphi_{n-1}^0\left(t\Lambda_k^{-1}\right) = \varphi_{n-1}^0\left(t\Lambda_k^{-1}\right) \ , \quad (5.27)$$

$$i = p-1: \ \ \varphi_{n}^1\left(t\Lambda_k^{-1}\right) = \varphi_{n}^0\left(t\Lambda_k^{-1}\right) - \varphi_{n-1}^0\left(t\Lambda_k^{-1}\right) \ , \quad (5.28)$$

$$i = p-2: \ \ \varphi_{n+1}^2\left(t\Lambda_k^{-1}\right) = \varphi_{n+1}^0\left(t\Lambda_k^{-1}\right) - \varphi_{n}^0\left(t\Lambda_k^{-1}\right) + \frac{1}{2}\varphi_{n-1}^0\left(t\Lambda_k^{-1}\right) \ . \quad (5.29)$$

One immediately sees that if we use the properties of the determinant and build linear combinations we can simplify the determinant kernel in Eq. (5.26) significantly. Adding Eq. (5.27) and Eq. (5.28) reduces Eq. (5.28) to the first term only. Adding the resulting expression of Eq. (5.28) and $-1/2$ times Eq. (5.27) to Eq. (5.29) leads to

$$i = p: \ \ \varphi_{n-1}^0\left(t\Lambda_k^{-1}\right) \ , \quad (5.30)$$

$$i = p-1: \ \varphi_n^0\left(t\Lambda_k^{-1}\right) \ , \tag{5.31}$$
$$i = p-2: \ \varphi_{n+1}^0\left(t\Lambda_k^{-1}\right) \ . \tag{5.32}$$

Continuing this iteratively with all rows in Eq. (5.26), we arrive at

$$E_p^{(2)}([0,t];p) = \frac{K_{p\times n} t^{pn-p(p-1)/2}}{\Delta_p(\Lambda)} \det\left[\varphi_{n-p+i-1}^0\left(t\Lambda_k^{-1}\right)\right] \ , \tag{5.33}$$

The coefficients of the rows when adding them mutually can be read off from Eq. (5.24). To show that Eq. (5.33) coincides with Eq. (5.14), we perform a change of integration variable in Eq. (5.22) and find

$$\varphi_{n-p+i-1}^0\left(t\Lambda_k^{-1}\right) = \left(\frac{1}{t}\right)^{n-p+i} \int\limits_0^t \mathrm{d}x \exp\left(-x/\Lambda_k\right) x^{n-p+i-1} \ . \tag{5.34}$$

Thus, inserting this into Eq. (5.33), taking the prefactor $(1/t)^{n-p+i}$ out of the determinant and employing the normalization condition $E_p^{(2)}([0,t];p) \to 1$ for $t \to \infty$ yields Eq. (5.14).

Regardless of the fact that an exact solution is known, we present a third way to solve the eigenvalue integral (5.15). It leads to an unknown expression for the gap probability. Similar to the previous approach we employ ideas from Ref. [111] and derive in appendix C.4 under the assumption that $p = 2l$ is even an expression with a determinant of dimension $l \times l$. It is given by

$$\begin{aligned} E_p^{(2)}([0,t];p) = &\frac{(-1)^{nl+l(l+1)/2} t^{np-l(l-1)} (2\pi)^{-l}}{\det^{n-l+1}\Lambda \Delta_l(\lambda)\Delta_l(\mu)(n-l)!} \prod_{j=0}^{n-l-1} \frac{(j+1)!}{(n-j-1)!} \\ \times &\det\left[\frac{(-1)^{n-l}(n-l)!\prod_{j=0}^{n-l-1}(n-j-1)! Z_{n-l+1}^{0/2}(\lambda_{k_1},\mu_{k_2})}{(2\pi)^{n-l}(n-l+1)!\prod_{j=0}^{n-l-1}(j+1)!}\right] \ , \end{aligned} \tag{5.35}$$

where $1 \leq k_1, k_2 \leq l$, $\Lambda = \mathrm{diag}(\lambda, \mu)$, with $\lambda_k = \Lambda_k$ and $\mu_k = \Lambda_{k+l}$ for all $k = 1, \ldots, l$. We introduce the two-point partition function $Z_{n-l+1}^{0/2}$, which is an average over an eigenvalue ensemble similar to Eq. (5.15), but with only two determinants in the denominator, see Eq. (C.33). We map it to a non-invariant Wishart model and compute it in appendix C.5, yielding

$$\begin{aligned} Z_{n-l+1}^{0/2}(\lambda_{k_1},\mu_{k_2}) = &\frac{(2\pi)^{n-l+1}\lambda_{k_1}\mu_{k_2}}{(n-l)!(n-l-1)!} \prod_{j=0}^{n-l} \frac{(j+1)!}{(n-1-j)!} \\ \times &\frac{\varphi_{n-l}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l-1}^{l-1}\left(\frac{t}{\mu_{k_2}}\right) - \varphi_{n-l-1}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l}^{l-1}\left(\frac{t}{\mu_{k_2}}\right)}{t(\lambda_{k_1}-\mu_{k_2})} \ . \end{aligned} \tag{5.36}$$

Substituting this expression for the two-point partition function into the gap probability (5.35), we arrive at a determinantal expression with an $l \times l$ dimensional matrix kernel,

$$
\begin{aligned}
E_p^{(2)}([0,t];p) &= \frac{2\pi(-1)^{l(l-1)/2} t^{np-l^2}}{\det^{n-l}\Lambda \Delta_l(\lambda)\Delta_l(\mu)(n-l+1)!^l (l-1)!^l} \\
&\times \det\left[\frac{\varphi_{n-l}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l-1}^{l-1}\left(\frac{t}{\mu_{k_2}}\right) - \varphi_{n-l-1}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l}^{l-1}\left(\frac{t}{\mu_{k_2}}\right)}{(\lambda_{k_1}-\mu_{k_2})}\right],
\end{aligned}
\tag{5.37}
$$

where $k_1, k_2 = 1, \ldots, l$. So far, we have discussed the even $p = 2l$ case only. If $p = 2l-1$ is odd, we derive a determinantal expression for the gap probability from Eq. (5.35). We introduce a dummy empirical eigenvalue $\mu_l = \Lambda_{2l}$ such that $p+1$ is even. The empirical eigenvalues are ordered into λ and μ such that λ includes the first l and μ the remaining $l-1$ empirical eigenvalues plus the dummy variable. We employ the analysis done to get from Eq. (5.15) to Eq. (5.37) and obtain an expression possessing a determinantal structure. Performing the limit $\mu_l \to 0$ in Eq. (5.37), we arrive at

$$
\begin{aligned}
E_p^{(2)}([0,t];p) &= \frac{2\pi(-1)^{l(l-1)/2} t^{np-l(l-1)}}{\det^{n-l}\Lambda \det\lambda \Delta_l(\lambda)\Delta_{l-1}(\mu)(n-l+1)!^{l-1} (l-1)!^l} \\
&\times \det\left[\begin{array}{c} \dfrac{\varphi_{n-l}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l-1}^{l-1}\left(\frac{t}{\mu_{k_2}}\right) - \varphi_{n-l-1}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right)\varphi_{n-l}^{l-1}\left(\frac{t}{\mu_{k_2}}\right)}{(\lambda_{k_1}-\mu_{k_2})} \\ \\ \varphi_{n-l}^{l-1}\left(\dfrac{t}{\lambda_{k_1}}\right) \end{array}\right],
\end{aligned}
\tag{5.38}
$$

where $1 \leq k_1 \leq l$ and $1 \leq k_2 \leq l-1$.

5.2.4 Quaternion Case

In the quaternion case, no closed-form expression for the unitary-symplectic Itzykson-Zuber integral is known. Thus, the only possibility to analyze the gap probability (2.37) is to use the eigenvalue integral representation obtained in section 5.2.1. We present two different solutions of Eq. (5.12). The expression obtained first, is given in terms of a $2n \times 2n$-dimensional Pfaffian. The second solution distinguishes between an even and an odd number p of empirical eigenvalues. For an even number of empirical eigenvalues $p = 2l$, we obtain an Pfaffian structure with a $p \times p$ dimensional kernel, whereas for odd $p = 2l-1$ it leads to a Pfaffian of dimension $(p+1) \times (p+1)$.

In section 5.2.1, we show that the probability to find all eigenvalues of a real quaternion Wishart matrix within the interval $[0,t]$ is given by

$$
E_p^{(4)}([0,t];p) = K t^{np2} \int d[Y] \; \frac{|\Delta_n(Y)|^4 \det^{-2n+1}(\imath Y + \mathbf{1}_n) \exp\left(2\mathrm{tr}\,(\imath Y + \mathbf{1}_n)\right)}{\prod_{k=1}^p \det^2\left(2t/\Lambda \mathbf{1}_n + \imath Y + \mathbf{1}_n\right)} \; .
\tag{5.39}
$$

The normalization constant is determined by the requirement that for $t \to \infty$ the gap probability $E_p^{(4)}([0,t];p) \to 1$. Applying it to Eq. (5.39), we find

$$K = \frac{2^{2pn-n}}{\det^{2n}\Lambda(2n)!(2\pi)^n} \ , \tag{5.40}$$

where we make use of Eq. (D.7). We approach the eigenvalue integral (5.39) as an averaged product of inverse characteristic polynomials. These averages have been extensively studied for all three ensembles in Refs. [107,110,112,173] and references therein. In appendix C.6, we adopt the results of Ref. [112] to the present case and find that the gap probability (5.39) is given by

$$E_p^{(4)}([0,t];p) = \frac{K\imath^{n^2}(-1)^{n+p(p+1)/2}n!t^{2np-p(p-1)/2}}{2^{p(p-1)/2}\det^{1-l}\Lambda\Delta_p(\Lambda)}\mathrm{pf}\left[\begin{array}{cc} A & B \\ -B^T & C \end{array}\right]. \tag{5.41}$$

We split the entries of the matrix in the Pfaffian determinant into four blocks. Performing the integrals within these blocks leads to

$$\begin{aligned}
\frac{2^{-2n+6}\Lambda_{k'}\Lambda_k}{(-\imath)2t\,(\Lambda_k-\Lambda_{k'})}A_{kk'} = {} & \frac{2\pi\imath\Lambda_{k'}^2\Lambda_k^2\ e^{-t/\Lambda_k}}{(\Lambda_k-\Lambda_{k'})^2}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-1}\right.\\
& + \left(-\frac{\Lambda_{k'}}{t}\right)^{2n-1} - (2n-1)\left(-\frac{\Lambda_k}{t}\right)^{2n} - (2n-1)\left(-\frac{\Lambda_{k'}}{t}\right)^{2n}\\
& \left. + \sum_{m=0}^{2n-2}\frac{(-1)^m(m+1)\ e^{t/\Lambda_k}}{(2n-2-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right]\\
& \frac{4\pi\imath\Lambda_{k'}^3\Lambda_k^3\ e^{-t/\Lambda_k}}{(\Lambda_k-\Lambda_{k'})^2}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-1} + \left(-\frac{\Lambda_{k'}}{t}\right)^{2n-1}\right.\\
& \left. + \sum_{m=0}^{2n-2}\frac{(-1)^m\ e^{t/\Lambda_k}}{(2n-2-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right] \ ,
\end{aligned} \tag{5.42}$$

for the upper left block,

$$\begin{aligned}
\frac{B_{lk}}{(-\imath)^2 2^{2n-l-1}} = {} & \frac{(l-1)t}{\Lambda_k}2\pi\imath\ e^{-t/\Lambda_k}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-l} - (2n-l)\left(-\frac{\Lambda_k}{t}\right)^{2n-l+1}\right.\\
& \left. + \sum_{m=0}^{2n-l}\frac{(-1)^m(m+1)\ e^{t/\Lambda_k}}{(2n-l-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right]\\
& + l2\pi\imath\ e^{-t/\Lambda_k}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-l-1} - (2n-l-1)\left(-\frac{\Lambda_k}{t}\right)^{2n-1}\right.\\
& \left. + \sum_{m=0}^{2n-l-1}\frac{(-1)^m(m+1)\ e^{t/\Lambda_k}}{(2n-l-1-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right]
\end{aligned} \tag{5.43}$$

for the off diagonal blocks and

$$\frac{2^{+l+l'+1-2n}C_{ll'}}{(-\imath)^2(l-l')} = \begin{cases} \dfrac{2\pi\imath}{(2n-l-l'+1)!} & , 2n+1 \geq l+l' \\ 0 & , \text{else} \end{cases} \tag{5.44}$$

for the lower right block. Since it is highly non-trivial to invert the matrix C, we can not use the Schur complement to reduce the dimension of the Pfaffian. Hence, the analytical considerations lead to a Pfaffian expression with $2n \times 2n$ dimensional matrix kernel. On account of $n \geq p$, this Pfaffian will be large and therefore challenging to evaluate even by a computer. Thus, it is reasonable to derive a more compact form.

To do so, we approach Eq. (5.39), in analogy to the third approach in the previous section. We introduce the weight

$$g(z,y) = \frac{\delta(y-z)}{(y-z)} \frac{\exp\left(2(\imath z+1)\right)}{(\imath z+1)^{2n-1}}, \tag{5.45}$$

construct a Pfaffian, apply the Schur complement and identify the elements of the matrix kernel with an appropriate eigenvalue integral. For technical reasons, we assume that $p = 2l$ is even. If $p = 2l-1$ is odd we use the results obtained for even p and perform a special limit. We show the detailed construction of the $2l \times 2l$ dimensional Pfaffian structure in appendix C.7. There we find that the gap probability (5.39) is given by

$$\begin{aligned} E_p^{(4)}([0,t];p) = {} & \frac{K\imath^{n^2}(-1)^{n+p(p+1)/2} n!\ t^{2np}}{(2t)^{p(p-1)/2}\det^{1-p}\Lambda\Delta_p(\Lambda)} \text{pf} M_d \\ & \times \text{pf}\left[K_4 \frac{t(\Lambda_k - \Lambda_{k'})}{\Lambda_k\Lambda_{k'}} Z_{n-l+1}^{0/2}\left(\Lambda_k, \Lambda_{k'}\right)\right] \ , \end{aligned} \tag{5.46}$$

where $1 \leq k, k', \leq 2l$, K_4 is a normalization constant readily determined by comparing Eq. (C.56) with Eq. (C.59) and

$$\text{pf} M_d = (2\pi)^{d/2} \frac{\imath^{(d/2)^2}(-1)^{d/2}}{(d/2)!} 2^{d(2n-d-1)/2} \prod_{j=0}^{d/2-1} \frac{(2j+2)!}{(2n-2-2j)!} \ . \tag{5.47}$$

The two-point partition function $Z_{n+1-l}^{0/2}$ introduced in Eq. (5.46) is similar to the gap probability, but with two empirical eigenvalues only. To compute it, we apply in appendix C.8 the analysis of section 5.2.1 in the backward direction and find

$$\begin{aligned} & \frac{Z_{n+1-l}^{0/2}\left(\Lambda_k, \Lambda_{k'}\right) t^2 \left(\Lambda_k - \Lambda_{k'}\right)^2}{(\Lambda_k\Lambda_{k'})^2 K_{n+1-l}} = -2\varphi_{2n-2l}^{2l-2}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi_{2n-2l}^{2l-2}\left(\frac{2t}{\Lambda_k}\right) \\ & + \varphi_{2n-2l+1}^{2l-2}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi_{2n-2l-1}^{2l-2}\left(\frac{2t}{\Lambda_k}\right) + \varphi_{2n-2l-1}^{2l-2}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi_{2n-2l+1}^{2l-2}\left(\frac{2t}{\Lambda_k}\right) \\ & - \frac{\varphi_{2n-2l}^{2l-2}\left(\frac{2t}{\Lambda_k}\right)\varphi_{2n-2l-1}^{2l-2}\left(\frac{2t}{\Lambda_{k'}}\right) - \varphi_{2n-2l-1}^{2l-2}\left(\frac{2t}{\Lambda_k}\right)\varphi_{2n-2l}^{2l-2}\left(\frac{2t}{\Lambda_{k'}}\right)}{t(\Lambda_k\Lambda_{k'})^{-1}\left(\Lambda_k - \Lambda_{k'}\right)} \\ & = \frac{t\left(\Lambda_k - \Lambda_{k'}\right)}{\left(\Lambda_k\Lambda_{k'}\right)} \ \Xi\left(\Lambda_k, \Lambda_{k'}\right) \ , \end{aligned} \tag{5.48}$$

where the normalization constant is given by

$$K_{n+1-l} = \frac{(1/2)^{(2n+1-2l)(n+2-l)}\,(2\pi)^{n-l+1}}{(2n-2l-1)!(2n-2l+1)!}\prod_{j=0}^{n-l}\frac{(2j+2)!}{(2n-2-2j)!}\ . \tag{5.49}$$

If we substitute Eq. (5.48) into the Pfaffian determinant (5.46) and take out all constants independent of k and k', we arrive at the following exact expression for the gap probability

$$E_p^{(4)}([0,t];p) = C_{n,p}\frac{t^{2np-p(p-1)/2}}{\det^{2n+1-p}\Lambda\Delta_p(\Lambda)}\mathrm{pf}\left[\Xi\left(\Lambda_k,\Lambda_{k'}\right)\right]\ , \tag{5.50}$$

where $1 \leq k, k' \leq p = 2l$ and

$$\begin{aligned} C_{n,p} = {} & \frac{(-1)^{l(l+1)/2+p(p+1)/2+n^2+l^2}n!}{(2n)!(n-l)!(2n-2l+1)!^l(2l-2)!^l}\prod_{j=0}^{n-l-1}\frac{(2j+2)!}{(2n-2-2j)!} \\ & \times 2^{2np-p(p-1)/2-n+2l(2-l(2n+2-l)+n(n+2))}\ . \end{aligned} \tag{5.51}$$

To derive the case of an odd number of empirical eigenvalues $p = 2l - 1$ from Eq. (5.50), we set Λ_{2l} to zero. Using the properties of the Pfaffian determinant, we take the limit into the Pfaffian and find

$$\begin{aligned} E_p^{(4)}([0,t];p) = {} & C_{n,p}\frac{t^{2n(p-1)-(l-1)(2l-1)}(2n-2l-1)!}{\det^{2n+1-2l}\Lambda\Delta_p(\Lambda)} \\ & \times \mathrm{pf}\begin{bmatrix} \Xi\left(\Lambda_k,\Lambda_{k'}\right) & \varphi_{2n-2l+1}^{2l-2}\left(\dfrac{2t}{\Lambda_k}\right) \\ -\varphi_{2n-2l+1}^{2l-2}\left(\dfrac{2t}{\Lambda_k}\right) & 0 \end{bmatrix}\ , \end{aligned} \tag{5.52}$$

where $1 \leq k, k' \leq 2l - 1$. Thus, for even and odd p it is possible to express the probability of finding all eigenvalues of a real quaternion Wishart matrix below t as a Pfaffian point process. This is surprising, because there is no reason to believe this by considering Eq. (2.37). There is even no reason to expect a Pfaffian structure at all.

5.2.5 Real Case

Because of half-integer powers of the characteristic polynomials in the denominator of the integrand (5.12), it not clear how to obtain a Pfaffian point process for the gap probability in the case of real correlated Wishart ensemble. By a Pfaffian point process we mean an Pfaffian structure such that each entry depends on two empirical eigenvalues only. This was the case for the real quaternion Wishart model discussed in the previous section.

Fortunately, we are able to show that the gap probability provides a Pfaffian structure. We include the square roots in a weight function,

$$w(y_i;\Lambda) = \frac{\exp\left(\imath y_i + 1\right)}{(\imath y_i+1)^{(n+1)/2}\prod_{k=1}^{p}\sqrt{\frac{t}{2}\Lambda_k^{-1}+\imath y_i+1}} \tag{5.53}$$

such that the computation reduces to the calculation of a partition sum with respect to the weight (5.53),

$$E_p^{(1)}([0,t];p) = Kt^{np/2} \int \mathrm{d}[Y] \ \ |\Delta_n(Y)| \prod_{i=1}^{n} w(y_i;\Lambda) \ . \tag{5.54}$$

The normalization constant is determined by the condition that for $t \to \infty$ the gap probability is one. Employing this in Eq. (5.54) and using (D.7) leads to

$$K = \frac{2^{n(p-4)/2}}{\det^{n/2}\Lambda \ \Gamma\left(\frac{n+2}{2}\right) \sqrt{\pi}^n} \ . \tag{5.55}$$

We apply the method of alternating variables [40] to Eq. (5.54) and find a Pfaffian determinant for the probability to find all eigenvalues below a certain threshold t. The expression obtained distinguishes between even and odd n. For an even number of eigenvalue integrals $n = 2L$ in Eq. (5.54) it is given by

$$E_p^{(1)}([0,t];p) = Kt^{2np} n! \mathrm{pf} \left[\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{y} \mathrm{d}y w(y;\Lambda) w(x;\Lambda) \left(x^{i-1}y^{j-1} - y^{i-1}x^{j-1}\right) \right] \ , \tag{5.56}$$

where $1 \leq i,j \leq 2L$. For odd $n = 2L+1$ it includes an extra row and column and reads

$$E_p^{(1)}([0,t];p) = Kt^{2np} n!$$
$$\mathrm{pf} \left[\begin{array}{cc} \int\limits_{-\infty}^{\infty} \mathrm{d}y \int\limits_{-\infty}^{y} \mathrm{d}y w(y;\Lambda) w(x;\Lambda) \left(x^{i-1}y^{j-1} - y^{i-1}x^{j-1}\right) & \int\limits_{-\infty}^{\infty} \mathrm{d}x w(x;\Lambda) x^{i-1} \\ -\int\limits_{-\infty}^{\infty} \mathrm{d}x w(x;\Lambda) x^{j-1} & 0 \end{array} \right] \ , \tag{5.57}$$

where i,j are as above. This is, compared to the real quaternion case not a Pfaffian point process, because each entry depends on all eigenvalues and the threshold parameter. Due to the complexity it is not clear how to perform the remaining twofold integral in the Pfaffian. We therefore leave the further investigation of it for future work.

5.2.6 Double Degenerate Approach

The gap probability (5.54) leads to an interesting observation. If we assume that $p = 2L$ is even and the empirical correlation matrix has an involution type of symmetry, *i.e.* it commutes with a matrix which has only two eigenvalues $L = \pm 1$ both L-fold degenerate, then C has a twofold degeneracy in the eigenvalue spectrum. In this case the square root singularities of Eq. (5.54) become ordinary poles such that a similar kind of analysis as for the complex and the real quaternion Wishart model is applicable. Although the resulting matrix model is simpler, this assumption is seldom justified empirically. However, we showed in section 4.4 that the bulk statistics of a Wishart model consisting of $lp \times ln$ dimensional data matrices with an l-fold degeneracy in the empirical eigenvalue spectrum, *i.e.* it has only p distinct empirical eigenvalues, approaches those of a Wishart model consisting of $p \times n$ data matrices without a degeneracy in the empirical eigenvalues. It is therefore reasonable to study Eq. (5.54) with double degeneracy in the empirical eigenvalues.

To begin, we consider the gap probability of a real correlated Wishart model consisting of $2p \times 2n$ dimensional data matrices. The empirical eigenvalues are supposed to be doubly degenerate such that only p distinct eigenvalues, ordered in $\Lambda = \text{diag}(\Lambda_1, \ldots, \Lambda_p)$, contribute. Thus, we want to compute

$$E_{2p}^{(1)}([0,t];2p) = Kt^{np2} \int \text{d}[Y] \, \frac{|\Delta_{2n}(Y)| \prod_{i=1}^{2n} w(y_i)}{\prod_{i=1}^{2n} \prod_{k=1}^{p} (t/(2\Lambda_k) + \imath y_i + 1)} \, , \tag{5.58}$$

where we introduce a new weight function,

$$w(y_i) = \frac{\exp\left(\imath y_i + 1\right)}{(\imath y_i + 1)^{(2n+1)/2}} \tag{5.59}$$

Although the asymptotic relation discussed earlier is valid for larger model correlation matrices only, the calculation of the gap probability (5.58) is from a mathematical point of view a large step forward. This is because even with degeneracy in the empirical eigenvalues the orthogonal Itzykson-Zuber integral (2.21) and therefore the joint probability distribution function (2.20) are highly non-trivial.

In appendix C.9, we apply standard techniques of random matrix theory and construct a Pfaffian expression for the gap probability (5.58) under the assumption of $p = 2l$. We obtain that it is given by

$$\begin{aligned} E_{p}^{(1)}([0,t];p) &= K_1 K t^{np2} (2n)! \text{pf} M_{2n-2l} \\ &\times \text{pf} \left[\mathcal{G}^{(1)}\left(\Lambda_k, \Lambda_{k'}\right) + \sum_{i,j=1}^{2n-2l} \mathcal{F}_j^{(1)}\left(\Lambda_k\right) (M_{2n-2l}^{-1})_{ji} \mathcal{F}_i^{(1)}\left(\Lambda_{k'}\right) \right] \, , \end{aligned} \tag{5.60}$$

where $1 \leq k, k' \leq 2l$,

$$\text{pf} M_{2n-2l} = \frac{(-1)^{n-l}(n-l)! \; 4^{2n-2l}}{(2n-2l)!} \prod_{j=0}^{2n-2l-1} \frac{\Gamma\left((j+3)/2\right)}{\Gamma\left((2n+1-j)/2\right)} \, , \tag{5.61}$$

$$D_{1,k}^{(1)}(y_{2i}) = \int\limits_{-\infty}^{y_{2i}} \frac{\text{d}x w_1(x)}{(t/(\Lambda_k 2) + 1 + \imath x)} \quad ; \quad D_{1,k}^{(2)}(y_{2i}) = \int\limits_{-\infty}^{y_{2i}} \text{d}x w_1(x) x^{j-1} \, , \tag{5.62}$$

and

$$[M_{2n-p}]_{ij} = \int\limits_{-\infty}^{\infty} \text{d}x \left(w(x) x^{i-1} D_{1,j}^{(2)}(x) - w(x) x^{j-1} D_{1,i}^{(2)}(x) \right) \, , \tag{5.63}$$

$$\mathcal{G}^{(1)}\left(\Lambda_k,\Lambda_{k'}\right)=\int\limits_{-\infty}^{\infty}\mathrm{d}x\left(\frac{w(x)D_{k'}^{(1)}(x)}{(\imath x+1+t/(\Lambda_k 2))}-\frac{w(x)D_{1,k}^{(1)}(x)}{(\imath x+1+t/(\Lambda_{k'}2))}\right)\ , \tag{5.64}$$

$$\mathcal{F}_j^{(1)}\left(\Lambda_k\right)=\int\limits_{-\infty}^{\infty}\mathrm{d}x\left(\frac{w(x)D_{1,j}^{(2)}(x)}{(\imath x+1+t/(\Lambda_k 2))}-x^{j-1}w(x)D_{1,k}^{(1)}(x)\right)\ . \tag{5.65}$$

Although the expression (5.60) is very compact, we are not able to compute all constitutes in the present basis. However, we can proceed differently and identify the kernel in Eq. (5.60) as suggested in Ref. [112] with an eigenvalue integral. To this end, we consider

$$Z_{2N}^{0/2}(\Lambda_k,\Lambda_{k'})=\int \mathrm{d}[Y]\ \frac{|\Delta_{2N}(Y)|\prod_{i=1}^{2N}w(y_i)}{\prod_{j=k,k'}\prod_{i=1}^{2N}(t/(2\Lambda_j)+\imath y_i+1)}\ , \tag{5.66}$$

and apply the analysis yielding Eq. (5.60), to solve it. Because it has only two empirical eigenvalues the Pfaffian obtained is trivial such that

$$\begin{aligned}Z_{2N}^{0/2}(\Lambda_k,\Lambda_{k'})=&\frac{\imath^{2N(2N-1)/2+(2N-2)(2N-3)/2}}{(\Lambda_k\Lambda_{k'})^{-1}(\Lambda_k-\Lambda_{k'})}\prod_{j=0}^{2N-2}\frac{\Gamma\left((j+3)/2\right)}{\Gamma\left((2n+1-j)/2\right)}\\ \times\frac{(2N)!4^{2N-2}}{(2N-2)!}&\left(\mathcal{G}^{(1)}\left(\Lambda_k,\Lambda_{k'}\right)+\sum_{i,j=1}^{2N-2}\mathcal{F}_j^{(1)}\left(\Lambda_k\right)(M_{2N-2}^{-1})_{ji}\mathcal{F}_i^{(1)}\left(\Lambda_{k'}\right)\right)\end{aligned} \tag{5.67}$$

Hence, if we set $N=n-l+1$, the Pfaffian kernel in Eq. (5.60) is proportional to the eigenvalue integral (5.66). We use it and construct a 4×4 real correlated Wishart model dual to Eq. (5.67). The two-point function (5.66) possesses a representation as real symmetric matrix model average,

$$Z_{2N}^{0/2}(\Lambda_k,\Lambda_{k'})\sim\int\frac{\mathrm{d}[H]\ \det^{-(2n+1)/2}(\imath H+\mathbf{1}_{2N})\exp\left(\mathrm{tr}\,(\imath H+\mathbf{1}_{2N})\right)}{\prod_{j=k,k'}\det(t/(2\Lambda_j)+\imath H+\mathbf{1}_{2N})}\ . \tag{5.68}$$

We apply section 5.2.1 backward and represent the determinant in the denominator of Eq. (5.68) as a Gaussian integral. If we exchange the Gaussian and the H integral and use Eq. (D.4), we find

$$\begin{aligned}Z_{2N}^{0/2}(\Lambda_k,\Lambda_{k'})\sim&\int\mathrm{d}[B]\ \exp\left(-\frac{t}{2}\mathrm{tr}BB^\dagger\mathrm{diag}(\Lambda_k\mathbf{1}_2,\Lambda_{k'}\mathbf{1}_2)\right)\\ &\times\det{}^{l-1}\left(\mathbf{1}_4-BB^\dagger\right)\Theta\left(\mathbf{1}_4-BB^\dagger\right)\ ,\end{aligned} \tag{5.69}$$

where B is a real $4\times(2n-2l-1)$ dimensional matrix. In contrast to the complex and real quaternion case, we can not use Eq. (5.69) to solve the kernel analytically. Because if we diagonalize $BB^\dagger$, the resulting orthogonal Itzykson-Zuber integral is unknown. On the other hand, if we are able to obtain an closed-form expression for the matrix kernel (5.67) by means of known functions, we have an expression for integrals of the form of Eq. (5.69).

We proceed differently as in the case of the complex and the real quaternion ensemble and study $\mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'})$, $\mathcal{F}_j^{(1)}(\Lambda_k)$ and $(M_{2N-2}^{-1})_{ij}$ separately. While $\mathcal{G}^{(1)}$ is analyzed using ordinary calculus, the remaining two quantities are computed with the aid of skew-orthogonal polynomials.

We begin our consideration with the derivation of $\mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'})$. The difficulty when doing calculations in the orthogonal ensembles is caused by the coupling of two integrals, *c.f.* Eq. (5.64). In appendix C.10 we introduce a Heaviside Θ-function to circumvent this difficulty and show that the coupled twofold integral reduces to a single integral

$$\begin{aligned}\mathcal{G}^{(1)}\left(\Lambda_k, \Lambda_{k'}\right) = &\frac{(-\imath)}{\Gamma^2((2n+1)/2)} \int\limits_{-1}^{1} d\tau \frac{(1-(\tau^-)^2)^{n-1/2}}{\tau - \imath} \\ &\times \left(\varphi_0^{n-1/2}\left(\frac{t(\tau^- + 1)}{2\Lambda_k}\right) \varphi_0^{n-1/2}\left(\frac{t(1-\tau^-)}{2\Lambda_{k'}}\right) - (\Lambda_k \leftrightarrow \Lambda_{k'}) \right) ,\end{aligned} \tag{5.70}$$

over a compact interval, where φ_s^m is for half-integer m introduced in Eq. (C.81). Although Eq. (5.70) is a transcendent function rather than a polynomial times an exponential, it is an important intermediate result. Because, similar to Eq. (5.66), we can map Eq. (5.64) to the following correlated Wishart model

$$\begin{aligned}\mathcal{G}^{(1)}\left(\Lambda_k, \Lambda_{k'}\right) \sim &\int d[B] \, \exp\left(-\frac{t}{2}\mathrm{tr}BB^\dagger \mathrm{diag}(\Lambda_k \mathbf{1}_2, \Lambda_{k'}\mathbf{1}_2)\right) \\ &\times \det{}^{(2n-1)/2}\left(\mathbf{1}_4 - BB^\dagger\right) \Theta\left(\mathbf{1}_4 - BB^\dagger\right) ,\end{aligned} \tag{5.71}$$

where B is now a real square 4×4 matrix. If we diagonalize $BB^\dagger = UXU$, where $X = \mathrm{diag}(x_1, x_2, x_3, x_4)$ and $U \in \mathrm{O}(4)$, the resulting average over $\mathrm{O}(4)$ reduces, in cause of $\mathrm{diag}(\Lambda_k \mathbf{1}_2, \Lambda_{k'}\mathbf{1}_2)$, to an integral over $\mathrm{O}(4)/\mathrm{O}(2) \times \mathrm{O}(2)$. This coset integral is not known in the literature. Hence, with Eq. (5.70) we provide a formula for Eq. (5.71) in terms of a transcendent function given as single integral.

In the reminder of this section, we are concerned with the computation of the $\mathcal{F}_j^{(1)}(\Lambda_k)$ and $[M_{2N-2}^{-1}]_{ij}$. We do this using the theory of skew-orthogonal polynomials. The sum

$$b_{2N-2}(\Lambda_k, \Lambda_{k'}) = \sum_{i,j=1}^{2N-2} \mathcal{F}_j^{(1)}\left(\Lambda_k\right)(M_{2N-2}^{-1})_{ji} \mathcal{F}_i^{(1)}\left(\Lambda_{k'}\right) \tag{5.72}$$

involved in the Pfaffian kernel (5.67) is a skew-symmertic bilinear form on a $2N-2$-dimensional vector space. The matrix defining the bilinear form is M_{2N-2}^{-1} and the vectors are $\mathcal{F}(x) = (\mathcal{F}_1(x), \ldots, \mathcal{F}_{2N-2}(x))$ with $x = \Lambda_k, \Lambda_{k'}$. The functions $\mathcal{F}_j^{(1)}$ aren't polynomials but projections of a analytic functions onto monomials x^{j-1},

$$\mathcal{F}_j^{(1)}\left(\Lambda_k\right) = \left\langle \frac{1}{\imath y + 1 + t/(2\Lambda_k)} \middle| x^{j-1} \right\rangle \tag{5.73}$$

where

$$\langle f|g\rangle = \int\limits_{-\infty}^{\infty} \mathrm{d}y \int\limits_{-\infty}^{y} \mathrm{d}x w(x)w(y)\left(f(y)g(x) - g(y)f(x)\right) \tag{5.74}$$

is a skew-symmetric scalar product with respect to the weight (5.59) on the space of polynomials. Since a bilinear form (5.72) is invariant under base changes, we can change the basis without changing its value. We choose it such that the monomial in the projection formula (5.73) becomes

$$x^{j-1} \mapsto R_{j-1}(x) \tag{5.75}$$

and find that

$$\mathcal{F}_j^{(1)}(\Lambda_k) \mapsto t_{j-1}(\Lambda_j) = \left\langle \frac{1}{\imath y + 1 + t/(2\Lambda_k)} \middle| R_{j-1}(x) \right\rangle . \tag{5.76}$$

Here $R_{j-1}(x)$ is a polynomial of degree $j-1$, skew-orthogonal with respect to the weight (5.59). This means that

$$\langle R_{2j+1}|R_{2i}\rangle = \epsilon_{ji} r_i \ , \quad \langle R_{2j+1}|R_{2i+1}\rangle = \langle R_{2j}|R_{2i}\rangle = 0 \ , \tag{5.77}$$

where $\epsilon_{ij} = \delta_{ij} = -\epsilon_{ji}$. According to Eq. (5.63), M_{2N-2} is the moment matrix with respect to the weight (5.59), i.e. $[M_{2N-2}]_{ij} = \langle y^{i-1}|x^{j-1}\rangle$. By the base transformation and the particular choice of polynomials, see Eq. (5.77), the moment matrix becomes

$$M_{2N-2} \mapsto \mathrm{diag}(r_0, \ldots, r_{N-1}) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{5.78}$$

If we express the bilinear form (5.72) in this basis, we arrive at

$$b_{2N-2}(\Lambda_k, \Lambda_{k'}) = \sum_{j=0}^{N-1} \frac{\left(t_{2j+1}\left(\frac{t}{2\Lambda_k}\right) t_{2j}\left(\frac{t}{2\Lambda_{k'}}\right) - t_{2j}\left(\frac{t}{2\Lambda_k}\right) t_{2j+1}\left(\frac{t}{2\Lambda_{k'}}\right)\right)}{r_j} \ , \tag{5.79}$$

and therefore reduces the computation of the bilinear form to the calculation of the projections t_j and the scalar products r_j. In the literature the former is sometimes referred to as the Cauchy transform of a skew-orthogonal polynomial.

The scalar products normalization r_j are derived as in section 3.1.2 using

$$r_j = \frac{Z_{2j+2}^{0/0}}{(2j+2)(2j+1) Z_{2j}^{0/0}}, \tag{5.80}$$

We read off the eigenvalue integral $Z_{2j+2}^{0/0}$ from Eq. (D.7) and substitute it into Eq. (5.80), leading to

$$r_j = \frac{8\pi (2j)! 2^{2n-4j}}{(2n-3-2j)!} \ . \tag{5.81}$$

To derive the Cauchy transform of the skew-orthogonal polynomials (5.73) we use the results of Ref. [174]. The authors show that these quantities have an expression in terms of an eigenvalue integral similar to Eq. (5.58) with one empirical eigenvalue only. We discuss this comprehensively in appendix C.11. There we show that the Cauchy transform of the polynomials with even degree are given by

$$\begin{aligned} t_{2j}(x) = -\frac{4\pi 2^{2n-2j}\imath^{2j+1}}{(2n-3-2j)!}\sum_{l_1,l_2=1}^{n-j-1}\begin{pmatrix} n-j-1 \\ l_1 \end{pmatrix}\begin{pmatrix} n-j-1 \\ l_2 \end{pmatrix} \\ \times\, x^{l_1+l_2+2j+2}\left(\psi_{l_1+1,l_2}(x)-\psi_{l_1,l_2+1}(x)\right)\ , \end{aligned} \tag{5.82}$$

whereas those for the polynomials of odd degree are given by

$$\begin{aligned} t_{2j+1}(x) = \frac{\imath t_{2j}(x)(x^2+l_1+l_2+2j+2)}{x} \\ -\frac{4\pi 2^{2n-2j}\imath^{2j}}{(2n-3-2j)!}\sum_{l_1,l_2=1}^{n-j-1}\begin{pmatrix} n-j-1 \\ l_1 \end{pmatrix}\begin{pmatrix} n-j-1 \\ l_2 \end{pmatrix} \\ +x^{(2j+1)/2+l_2}\exp(-x)\left(\Gamma(l_1+(2j+3)/2)-\Gamma(l_1+(2j+3)/2;x)\right) \\ -\, x^{(2j+3)/2+l_2}\exp(-x)\left(\Gamma(l_1+(2j+1)/2)-\Gamma(l_1+(2j+1)/2;x)\right)\Big] \end{aligned} \tag{5.83}$$

with

$$\begin{aligned} \psi_{\alpha_1,\alpha_2}(x) = \int_0^x \mathrm{d}x x^{(2j-1)/2+\alpha_2}\exp(-x) \\ \times\left(\Gamma(\alpha_1+(2j+1)/2)-\Gamma(\alpha_1+(2j+1)/2;x)\right) \end{aligned} \tag{5.84}$$

and $\Gamma(m;x)$ the incomplete Γ-function [136]. Although the final constitutes t_{2j} and t_{2j+1} of b_{2N+1} are rather cumbersome and involve an complicated one-dimensional integral, they can be efficiently implemented into a computer program to finally evaluate the Pfaffian kernel. Furthermore, we are able to circumvent the highly non-trivial group integral involved in Eq. (5.69) when diagonalizing $BB^\dagger$. This observation suggests that when considering eigenvalue statistics in the correlated Wishart ensemble, the entire form of the orthogonal Itzykson-Zuber integral is not important, but only the eigenvalue integrals involving it.

Hence, combining Eq. (5.60) with Eq. (5.70) and Eq. (5.79), the gap probability that all eigenvalues of a real correlated Wishart matrix lie below a threshold t is given by

$$E_p^{(1)}([0,t];p) = K_1 K t^{np2}(2n)!\,\mathrm{pf} M_{2n-2l}\,\mathrm{pf}\left[\Xi(\Lambda_k,\Lambda_{k'})\right]\ , \tag{5.85}$$

where

$$
\begin{aligned}
\Xi(\Lambda_k,\Lambda_{k'}) = {} & \frac{4\pi^2(-\imath)}{\Gamma^2((2n+1)/2)}\int\limits_{-1}^{1}\mathrm{d}\tau\frac{(1-(\tau^-)^2)^{n-1/2}}{\tau-\imath} \\
& \times\left(\varphi_0^{n-1/2}\left(\frac{t(\tau^-+1)}{2\Lambda_k}\right)\varphi_0^{n-1/2}\left(\frac{t(1-\tau^-)}{2\Lambda_{k'}}\right)-(\Lambda_k\leftrightarrow\Lambda_{k'})\right) \\
& +\sum_{j=0}^{n-l}\frac{\left(t_{2j+1}\left(\frac{t}{2\Lambda_k}\right)t_{2j}\left(\frac{t}{2\Lambda_{k'}}\right)-t_{2j}\left(\frac{t}{2\Lambda_k}\right)t_{2j+1}\left(\frac{t}{2\Lambda_{k'}}\right)\right)}{r_j}\ ,
\end{aligned}
\tag{5.86}
$$

and $1\leq k,k'\leq p$. For odd $p=2l-1$ we insert into Eq. (5.58) a dummy empirical eigenvalue Λ_{2l} and apply the analysis for even p leading to the Pfaffian structure (5.60). If we take the limit $\Lambda_{2l}\to 0$, we arrive at

$$
\begin{aligned}
E_p^{(1)}([0,t];p) = {} & K_1Kt^{np2}(2n)!\frac{(2n-2l)!(n-l+1)!(4\pi)^{n-l+1}16}{(2n-2l+2)!(l-1)!t^{2n-2l+2}} \\
& \times\,\mathrm{pf}M_{2n-2l}\ \mathrm{pf}\left[\begin{array}{cc}\Xi(\Lambda_k,\Lambda_{k'}) & \dfrac{t_{2n-2l}\left(t/(2\Lambda_{k'})\right)}{r_{n-l}} \\ -\dfrac{t_{2n-2l}\left(t/(2\Lambda_k)\right)}{r_{n-l}} & 0\end{array}\right]\ ,
\end{aligned}
\tag{5.87}
$$

where $1\leq k,k'\leq p-1$. Thus, for even as well as for odd p we are able to derive a Pfaffian expression for $E_p^{(1)}([0,t];p)$. We reduced the matrix kernel to sums including an integral, which to the best of our knowledge, does not have a closed-form expression in terms of known functions. Apart from this, the gap probabilities (5.85) and (5.87) show that for a twofold degeneracy in the empirical eigenvalues, the orthogonal Itzykson-Zuber integral can be circumvented.

5.3 Smallest Eigenvalue Statistics

To discuss the limiting statistics of the smallest eigenvalue of a correlated Wishart matrix in section 5.4, we apply the Fourier approach as introduced in section 5.1 to the gap probability (2.44). Since we computed an exact expression as well as analyzed the microscopic limit of it in section 4.1, we focus in this section on the construction of an eigenvalue integral representation for the gap probability in a unified approach for $\beta=1,2,4$.

For the probability to find all eigenvalues above a threshold s, we derived a Wishart matrix model in section 2.3.3,

$$
E_p^{(\beta)}([0,s];0)=Ks^{np\beta/2}\int\mathrm{d}[W]\ P(W|s\hat{\Lambda}^{-1})\Theta(WW^\dagger-\mathbf{1}_{\gamma_2p})\ . \tag{5.88}
$$

Analogous to section 5.2.2, we would like to apply the Fourier approach and to take the Θ-function as observable. However, this is not possible, because $\Theta(WW^\dagger-\mathbf{1}_{\gamma_2p})\neq\Theta(W^\dagger W-\mathbf{1}_{\gamma_2n})=0$. To see this, we compare the eigenvalues of both matrices. The former, $WW^\dagger - \mathbf{1}_{\gamma_2 p}$, has p eigenvalues, each of which coincides with an eigenvalue of $W^\dagger W - \mathbf{1}_{\gamma_2 n}$. The remaining $n-p$ of the latter are -1. Since these are negative, the Heaviside Θ-function of $W^\dagger W - \mathbf{1}_{\gamma_2 n}$ is always zero. To apply the analysis leading to Eq. (5.3), we perform two successive changes of coordinates. We set $A = WW^\dagger$, inducing a transformation of the volume form given by

$$\mathrm{d}[W] \mapsto \det{}^{(n-p+1-2/\beta)/\gamma_1} A\ \Theta(A)\mathrm{d}[A] \ , \tag{5.89}$$

where A is real symmetric, Hermitian or real quaternion self-dual for $\beta = 1, 2, 4$. Subsequently, we substitute $A = \widetilde{W}\widetilde{W}^\dagger$, where $\widetilde{W} \in \mathrm{M}_{p\times p}(\mathbb{K})$, inducing another transformation of the volume form,

$$\mathrm{d}[A] \mapsto \det{}^{(2/\beta-1)/\gamma_1}\widetilde{W}\widetilde{W}^\dagger \mathrm{d}[\widetilde{W}] \ . \tag{5.90}$$

Applying both coordinate changes successively in Eq. (5.88) leads to a Wishart model consisting of $p \times p$ data matrices,

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K s^{np\beta/2} \int \mathrm{d}[\widetilde{W}]\ \det{}^{(n-p)/\gamma_1}\widetilde{W}\widetilde{W}^\dagger\ P(\widetilde{W}|s\hat{\Lambda}^{-1}) \\ \times\, \Theta(\widetilde{W}\widetilde{W}^\dagger - \mathbf{1}_{\gamma_2 p}) \ . \end{aligned} \tag{5.91}$$

As a consequence, all p eigenvalues of $\widetilde{W}\widetilde{W}^\dagger - \mathbf{1}_{\gamma_2 p}$ coincide with those of $\widetilde{W}^\dagger\widetilde{W} - \mathbf{1}_{\gamma_2 p}$. Thus, because of the two transformations, we are able to replace in our observable, the Θ-function, $\widetilde{W}\widetilde{W}^\dagger$ by $\widetilde{W}^\dagger\widetilde{W}$ without changing the integral. We insert an integral over a δ-function into Eq. (5.91), replace the matrix $\widetilde{W}^\dagger\widetilde{W}$ by a matrix Q and arrive at

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K s^{np\beta/2} \int \mathrm{d}[\widetilde{W}] \int \mathrm{d}[Q]\ \det{}^{(n-p)/\gamma_1} Q P(\widetilde{W}|s\hat{\Lambda}^{-1}) \\ \times\, \delta(Q - \widetilde{W}^\dagger\widetilde{W})\Theta(Q - \mathbf{1}_{\gamma_2 p}) \ , \end{aligned} \tag{5.92}$$

where Q is in the same symmetry class as $\widetilde{W}^\dagger\widetilde{W}$. As explained in section 5.1, we express the δ-function as a regularized Hermitian matrix integral and perform the $\widetilde{W}$ integral. By the exchange of the integrals, the $\widetilde{W}$ integral becomes a Gaussian. We perform it and are left with

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K \int \mathrm{d}[H]\ \frac{\exp\left(\mathrm{tr}\,(\imath H - \epsilon\mathbf{1}_{p\gamma_2})\right)}{\det^{1/\gamma_2}\left(\mathbf{1}_{\gamma_2 p^2} + \frac{2}{s\beta}\Lambda\otimes(\imath H - \epsilon\mathbf{1}_{p\gamma_2})\right)} \\ \times \int \mathrm{d}[Q]\ \Theta(Q)\det{}^{(n-p)/\gamma_1}(Q + \mathbf{1}_{p\gamma_2})\exp\left(\mathrm{tr}\,(\imath H - \epsilon\mathbf{1}_{p\gamma_2})\,Q\right) \ , \end{aligned} \tag{5.93}$$

where H is in the same symmetry class as Q and $0 < \epsilon < 1/\mathrm{max}(\Lambda_k)$. Since we decoupled Λ from H, the integral (5.93) is invariant under the action of G_p given by $H \mapsto UHU^\dagger$. Thus, we diagonalize H and obtain

$$\begin{aligned} E_p^{(\beta)}([0,s];0) = K \int \mathrm{d}[Y]\ \frac{|\Delta_p(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\,(\imath Y - \epsilon\mathbf{1}_p)\right)}{\det^{\beta/2}\left(\mathbf{1}_{p^2} + \frac{2}{s\beta}\Lambda\otimes(\imath Y - \epsilon\mathbf{1}_p)\right)} \\ \times \int \mathrm{d}[Q]\ \Theta(Q)\det{}^{(n-p)/\gamma_1}(Q + \mathbf{1}_{p\gamma_2})\exp\left(\mathrm{tr}\,(\imath Y\otimes\mathbf{1}_{\gamma_2} - \epsilon\mathbf{1}_{\gamma_2 p}))Q\right) \ . \end{aligned} \tag{5.94}$$

To compute the remaining Q integral, we can use the Sekiguchi differential operator and a modification of statement 5.1 of Ref. [113]. Because the analysis in the next section is concerned with the asymptotic behavior of the determinant in the denominator of the first line in Eq. (5.94) only, we stick to the representation of the gap probability (5.94) including the Q integral.

5.4 Large Wishart Correlation Matrices

In many situations in data analysis one is confronted with large correlation matrices, *i.e.* the dimension of the matrix tend to infinity. Since Wishart model correlation matrices consist of rectangular data matrices, two natural limiting regimes, the hard edge and the soft edge limit, are considered, *c.f.* section 2.2.2.

We present an analysis of the extreme eigenvalue statistics in the soft edge scaling, *i.e.* $n, p \to \infty$ and $p/n = \gamma^2 < 1$ fixed and of the largest eigenvalue statistics in the hard edge scaling, *i.e.* $n, p \to \infty$ and $n - p = \nu$ fixed. We show that in some cases the limiting distribution of the smallest and the largest eigenvalue carry over from the uncorrelated to the correlated Wishart model.

In section 5.4.1 and 5.4.2, we study the gap probabilities (2.37) and (2.43) related to the distribution of the largest and the smallest eigenvalue, respectively. We show that within the real, the complex and the real quaternion correlated Wishart model both quantities converge to the Tracy-Widom distribution, in some cases. We confirm our findings in section 5.4.3 with numerical simulations.

5.4.1 Limiting Largest Eigenvalue Statistics

To study the limiting distribution of the smallest and largest eigenvalue, we summarize known results for the uncorrelated Wishart model, which we later use to derive our main statements.

For $\Lambda = \mathbf{1}_p$, the Itzykson-Zuber integral (2.21) present in Eq. (2.20) is trivial such that the joint eigenvalue distribution function is given by

$$P(X) \sim |\Delta_p(X)|^\beta \prod_{i=1}^{p} x_i^{\beta(n-p+1-2/\beta)/2} \exp\left(-\beta x_i/2\right) = |\Delta_p(X)|^\beta \prod_{i=1}^{p} w(x_i) \qquad (5.95)$$

To analyze the statistics of the largest eigenvalue, we use the gap probability (2.37) and evaluate it at the upper edge of the spectrum on the scale of the typical fluctuations of the largest eigenvalue. This is equivalent to appropriately centering and rescaling all eigenvalues,

$$\frac{x_i - \mu_+}{\sigma_+} \ , \qquad (5.96)$$

where σ_+ and μ_+ are yet to be determined. Following Ref. [91], these constants are fixed using the "log gas" method [44], which is known to mathematicians as Stieltjes "electrostatic interpretation" [175]. It relates the zeros of the weighted Laguerre polynomials

$$\phi_n^\alpha(x) = w(x) L_n^{(\alpha)}(x) \ , \qquad (5.97)$$

where $w(x)$ was introduced in Eq. (5.95), to the location of the eigenvalues of a large Wishart matrix. In this picture, the largest eigenvalue corresponds to the largest root of the weighted Laguerre polynomial, *i.e.* it corresponds to an value of $x \sim \mathcal{O}(n)$. If the Laguerre polynomial is coincidentally studied for a large argument x and degree n its asymptotic behavior is mimicked by an Airy function Ai(x) with zero boundary conditions [175]. This argument was used in Ref. [91] to determine the scaling for the complex uncorrelated Wishart model to be

$$\mu = 4p + 2(n-p) + 2 \quad \text{and} \quad \sigma = 2(2p/3)^{1/3} \tag{5.98}$$

whenever the rectangularity $n-p$ is kept fixed. If the eigenvalues are centered and rescaled according to Eq. (5.98), the two point function $R_2(x,y)$ becomes the famous Airy kernel [91].

It is well known [40, 44, 176] that gap probabilities in general, and therefore the gap probability corresponding to the largest eigenvalue $E_p^{(2)}([0,t];p)$, can be written in terms of Fredholm determinants of particular integral operators in L^2 space [40,44]. For the scaling (5.98), the integral kernels at the edge of the spectrum are the Airy kernels. For the GUE the authors in Ref. [177, 178] constructed a completely integrable system of differential equations for Fredholm determinants with Airy kernels and showed that the gap probability of the largest eigenvalue becomes the cumulative density function, $F_2(\chi)$, of the Tracy-Widom distribution, $f_2(\chi) = \mathrm{d}F_2(\chi)/\mathrm{d}\chi$. Together with the analysis of the edge statistics in Ref. [91] this yields for the complex uncorrelated Wishart model,

$$E_p^{(2)}([0,\mu+\sigma\chi];p) \to F_2(\chi) = \exp\left(-\int_{\chi}^{\infty}(x-\chi)q^2(x)\mathrm{d}x\right) , \tag{5.99}$$

for $n,p \to \infty$ and p/n fixed, where $q(x)$ is the solution to the Painlevé II equation [179]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}q(x) = xq(x) + 2q^3(x) \tag{5.100}$$

with respect to the large $x \to \infty$ behavior $q(x) \sim$ Ai(x). Later, it was shown in Ref. [180] using a different scaling,

$$\mu_+ = (\sqrt{p}+\sqrt{n})^2 \quad \sigma_+ = (\sqrt{p}+\sqrt{n})\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{n}}\right)^{1/3} , \tag{5.101}$$

that the limit (5.99) holds even for $n-p \sim \mathcal{O}(n)$. This result was extended to the real uncorrelated Wishart model in Ref. [92]. The author proved that the largest eigenvalue distribution converges in the limit $n,p \to \infty$ and $n-p \sim \mathcal{O}(n)$ to the Tracy-Widom distribution $f_1(\chi) = \mathrm{d}F_1(\chi)/\mathrm{d}\chi$. For $\beta = 1,4$ the corresponding limiting cumulative density functions F_1 and F_4 are related to F_2 by [181]

$$F_1^2(\chi) = F_2(\chi)\exp\left(-\int_{\chi}^{\infty}q(x)\mathrm{d}x\right) \tag{5.102}$$

and

$$F_4^2\left(\frac{\chi}{2^{2/3}}\right) = F_2(\chi)\cosh^2\left(-\frac{1}{2}\int\limits_{\chi}^{\infty} q(x)\mathrm{d}x\right) \ , \tag{5.103}$$

respectively. The scaling used in Ref. [92] to show that the largest eigenvalue in the uncorrelated real Wishart model is the Tracy-Widom distribution,

$$\mu_{p,n} = (\sqrt{p}+\sqrt{n_-})^2 \quad \sigma_{p,n} = (\sqrt{p}+\sqrt{n_-})\left(\frac{1}{\sqrt{p}}+\frac{1}{\sqrt{n_-}}\right)^{1/3} \ , \tag{5.104}$$

where $n_- = n-1/2$, is asymptotically the same as Eq. (5.101), but the former yields a better convergence. The results of Ref. [92] hold as well for $\nu = n-p$ fixed.

Yet all approaches were concerned with Gaussian distributed data matrices, Eq. (2.16). In Ref. [182] the author considered the statistics of the largest eigenvalue for correlation matrices consisting of (non-Gaussian distributed) Wigner matrices. He showed that the largest as well as the second, third., etc. largest eigenvalue, follow the Tracy-Widon distribution after some proper rescaling.

For the ensemble of uncorrelated real quaternion Wishart matrices, the limiting distribution of the largest eigenvalue was studied among other related things in Ref. [99]. The author proved the convergence of the cumulative density function of the centered and rescaled largest eigenvalue to F_4, see Eq. (5.103). The centering and rescaling parameters are the same as those used in Ref. [180], see Eq. (5.101).

We now turn from summarizing the results about the largest eigenvalue statistics within the uncorrelated Wishart model to the results within the correlated Wishart model. In Ref. [92] the author proposes the spiked population model, where the empirical eigenvalues are chosen to be $\Lambda_p \neq \Lambda_{p-1} \neq \cdots \Lambda_{p-M+1} \neq \Lambda_{p-M} = \Lambda_{p-M+1} = \cdots = \Lambda_1$ and M finite as n, p tend to infinity. This means only a finite number of them differs from each other. If $\Lambda_p, \ldots, \Lambda_{p-M+1}$, for $p/n = \gamma^2$ fixed and $n, p \to \infty$, are smaller than the upper edge of the Marčenko-Pastur distribution (4.141), it is shown for the complex correlated Wishart model that the largest eigenvalue is Tracy-Widom (5.99) distributed [183]. When one eigenvalue is larger than the upper edge, it separates from the bulk of the spectrum and becomes Gaussian distributed with fluctuations of order $\sqrt{n}$ [37, 82]. Furthermore, in Ref. [184] the authors obtain a phase transition between these regimes and propose that this will hold with different limiting distributions also in the real spiked Wishart case. The conjecture has been partially proven for $M = 1$ in the real quaternion [99] and the real Wishart model [98, 170].

The general situation where all eigenvalues Λ_k are different, is considered for the complex ensemble in Ref. [160]. Under modest assumptions about the distribution of the empirical eigenvalues, it is shown that the limiting distribution of the largest eigenvalue is $f_2(\chi)$.

We study the gap probability (5.12) for n, p tending to infinity. Since there is no difference in the argumentation, we will not comment in the following on the two different regimes $n-p$ or $p/n = \gamma^2$ fixed, during the calculation. In section 5.2.1, we show that the probability of finding all eigenvalues in the real, complex, real quaternion correlated Wishart model below a threshold t is given by

$$E_p^{(\beta)}([0,t];p) = K \int \frac{\mathrm{d}[Y] \ |\Delta_n(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\left(\imath Y + \mathbf{1}_n\right)\right)}{\det^{\alpha\beta/2}\left(\imath Y + \mathbf{1}_n\right) \prod\limits_{k,i=1}^{p,n} \left(1 + \left(\imath y_i + 1\right) 2\Lambda_k/(t\beta)\right)^{\beta/2}} \ , \tag{5.105}$$

where the normalization constant K is yet to be determined and $\alpha = n - 1 + 2/\beta$. We assume a scaling of the threshold parameter of the form $t = \mu_+(\Lambda) + \sigma_+(\Lambda)\chi$, where both $\sigma_+(\Lambda)$ and $\mu_+(\Lambda)$ are assumed to be large, *i.e.* scale with p. We will justify this assumption later. A clue that it holds is due to the uncorrelated case, where $\Lambda_k = 1$ for all k and σ_+ and μ_+ are given by Eq. (5.101). Assuming that the empirical eigenvalues Λ_k are of order $\mathcal{O}(1)$ for all k and n, p tending to infinity, leads to the following estimate

$$p^{1-m}\Lambda_{\mathrm{min}}^m \leq \frac{1}{p^m}\left\langle \Lambda^m \right\rangle_{\mathrm{s}} \leq p^{1-m}\Lambda_{\mathrm{max}}^m \ . \tag{5.106}$$

where we introduce the sample average $\langle\cdot\rangle_{\mathrm{s}} = p^{-1}\mathrm{tr}\,(\cdot)$. For the class of empirical correlation matrices C with this property, only the rescaled trace with $m=1$ does not tend to zero when $p \to \infty$. Moreover, a simple estimate shows that for all m

$$\left\langle \Lambda^m \right\rangle_{\mathrm{s}} \sim \mathcal{O}(1) \ , \tag{5.107}$$

such that we are not able to estimate the asymptotic behavior of the variance in the empirical eigenvalues. A valid assumption for the sample variance of the empirical eigenvalues is

$$\mathrm{Var}_{\mathrm{s}}(\Lambda) = \left\langle \Lambda^2 \right\rangle_{\mathrm{s}} - \left\langle \Lambda \right\rangle_{\mathrm{s}}^2 \sim \ \mathcal{O}(1/p^c), \tag{5.108}$$

where $c > 0$ is a free parameter which is fixed later. To understand this assumption we use the Tschebyscheff inequality stating that

$$\mathbb{P}(|\Lambda - \langle\Lambda\rangle_{\mathrm{s}}| \geq x) \leq \frac{\mathrm{Var}_{\mathrm{s}}(\Lambda)}{x^2} \sim \frac{1}{x^2}\mathcal{O}(1/p^c) \ . \tag{5.109}$$

This inequality can be interpreted as follows. The larger the set of the empirical eigenvalues becomes, the smaller the distance between the smallest and largest empirical eigenvalue becomes. We employ the Tschebyscheff inequality (5.109) and approach the empirical eigenvalues as follows

$$\Lambda_k = \bar{\Lambda} + p^{-c}\Lambda_k^{(1)}, \tag{5.110}$$

where $\bar{\Lambda} = \langle\Lambda\rangle_{\mathrm{s}}$ and $\Lambda_k^{(1)} \sim \mathcal{O}(1)$. If C is a "real" correlation matrix properly normalized, then $\bar{\Lambda} = \langle\Lambda\rangle_{\mathrm{s}} = 1$. If we substitute Eq. (5.110) into Eq. (5.105), we are left with estimating the large p, n behavior of the np-fold product in the denominator of Eq. (5.105). Under the assumption that t is large, we find for each of the n individual factors in the product

$$\prod_{k=1}^{p} \frac{1}{\left(1+(\imath y_i+1)\,\frac{2}{t\beta}\left(\bar{\Lambda}+p^{-c}\Lambda_k^{(1)}\right)\right)^{\beta/2}} = \frac{1}{\left(1+(\imath y_i+1)\,\frac{2\bar{\Lambda}}{t\beta}\right)^{p\beta/2}}\left(1-\frac{\mathrm{tr}\Lambda^{(1)}}{p^c}\frac{(\imath y_i+1)\,t^{-1}}{\left(1+(\imath y_i+1)\,\frac{2\bar{\Lambda}}{t\beta}\right)^{\beta/2+1}}+\mathcal{O}(p^{-2c})\right)\,, \tag{5.111}$$

where $i=1,\ldots,n$. Inserting the expansion (5.111) into the gap probability Eq. (5.105) and keeping only the leading terms in n and p, we arrive at

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = {} & K\int \mathrm{d}[Y]\ \frac{|\Delta_n(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\,(\imath Y+\mathbf{1}_n)\right)}{\det^{\alpha\beta/2}(\imath Y+\mathbf{1}_n)\det^{p\beta/2}\left(\mathbf{1}_n+\frac{2\bar{\Lambda}}{t\beta}(\imath Y+\mathbf{1}_n)\right)} \\ & -\frac{\mathrm{tr}\Lambda^{(1)}}{p^c p\bar{\Lambda}}t\frac{\mathrm{d}}{\mathrm{d}t}K\int \mathrm{d}[Y]\ \frac{|\Delta_n(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\,(\imath Y+\mathbf{1}_n)\right)}{\det^{\alpha\beta/2}(\imath Y+\mathbf{1}_n)\det^{p\beta/2}\left(\mathbf{1}_n+\frac{2\bar{\Lambda}}{t\beta}(\imath Y+\mathbf{1}_n)\right)} \\ & +\mathcal{O}(p^{-2c})\,, \end{aligned} \tag{5.112}$$

The term in the first line of Eq. (5.112) is the gap probability that all eigenvalues of an uncorrelated Wishart matrix with variance $\bar{\Lambda}$ lie above a certain threshold t. From the discussion at the beginning of this section it turns out that if $\mu_+(\Lambda)=\bar{\Lambda}\mu_+$, $\sigma_+(\Lambda)=\bar{\Lambda}\sigma_+$, where μ_+ and σ_+ are as in Eq. (5.101), the first term on the right hand side of Eq. (5.112) converges to $F_\beta(\chi)$. If the second and all higher order terms tend to zero for n,p tending to infinity, we have shown that the distribution of the largest eigenvalue is the Tracy-Widom distribution.

The first correction term in the second line of Eq. (5.112) and all other corrections are powers of the derivative $p^{-c}\ t\mathrm{d}/\mathrm{d}t$ of an order $\mathcal{O}(1)$ function, because $p^{-1}\mathrm{tr}\Lambda^{(1)}\to$ const. and

$$K\int \mathrm{d}[Y]\ \frac{|\Delta_n(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\,(\imath Y+\mathbf{1}_n)\right)}{\det^{\alpha\beta/2}(\imath Y+\mathbf{1}_n)\det^{p\beta/2}\left(\mathbf{1}_n+\frac{2\bar{\Lambda}}{t\beta}(\imath Y+\mathbf{1}_n)\right)} \to F_\beta(\chi), \tag{5.113}$$

for $p\to\infty$ and p/n fixed. Thus we are left with analyzing the scaling behavior of the derivative induced by the centered and rescaled threshold parameter $t=\mu_++\sigma_+\chi$. Expressing the derivative in terms of a derivative with respect to χ yields

$$\left(p^{-c}t\frac{\mathrm{d}}{\mathrm{d}t}\right)^m = \left(\frac{\mu_++\sigma_+\chi}{p^c\sigma_+}\frac{\mathrm{d}}{\mathrm{d}\chi}\right)^m = \frac{\mu_+^m}{p^{mc}\sigma_+^m}\frac{\mathrm{d}^m}{\mathrm{d}\chi^m}+\mathcal{O}\left(\frac{\mu_+^{m-1}}{p^{c(m-1)}\sigma_+^{m-1}}\right), \tag{5.114}$$

because χ is of order $\mathcal{O}(1)$. Substituting Eq. (5.101) into the leading order term in p on the left hand side of Eq. (5.114) leads for $m=1$ to

$$\frac{1}{p^c}\frac{\mu_+}{\sigma_+} = \gamma^{2c-1}(1+\gamma)^{2/3}n^{2/3-c}\,. \tag{5.115}$$

Thus, if we require that $c > 2/3$, the second term in Eq. (5.112) is of order $\mathcal{O}(n^{2/3-c})$ and therefore goes to zero when n and p tend to infinity. The same holds for all higher order corrections in Eq. (5.112) such that $E_p^{(\beta)}([0,t];p)$ converges to $F_\beta(\chi)$.

This observation is valid for all regimes discussed above, because in any case $\mu_+ \sim \mathcal{O}(\text{max}(n,p))$ and $\sigma_+ \sim \mathcal{O}(\text{max}(n,p)^{1/3})$. The scaling requirement for the "fluctuations" of the empirical eigenvalues (5.108) leads to a macroscopic distance between the fluctuations of the largest eigenvalue and those of empirical eigenvalues. Thus, if the variance of the empirical eigenvalues is small enough, the empirical eigenvalues do not influence the statistics of the largest eigenvalue and therefore decouple from its statistics. If we combine the above arguments we can make the following statement.

Statement*: Let Λ be the matrix of distinct eigenvalues of the empirical correlation matrix C. The eigenvalues are chosen such that they are all of order $\mathcal{O}(1)$ with variance of order $\mathcal{O}(p^{-c})$ and $c > 2/3$. If n, p tend to infinity while $p/n \leq 1$ is fixed, then we obtain that*

$$E_p^{(\beta)}([0, \mu_+(\Lambda) + \sigma_+(\Lambda)\chi]; p) \to F_\beta(\chi) \ , \tag{5.116}$$

where $\mu_+(\Lambda) = \mu_+\bar{\Lambda}$, $\sigma_+(\Lambda) = \sigma_+\bar{\Lambda}$ and μ_+, σ_+ as introduced in Eq. (5.101) for $\beta = 1, 2, 4$ and $F_\beta(\chi)$ the cumulative density function shown in Eqs. (5.99), (5.102) and (5.103).

5.4.2 Limiting Smallest Eigenvalue Statistics

In this section, we consider the statistics of the smallest eigenvalue in the correlated Wishart model in a similar fashion as we approach the largest eigenvalue statistics in the previous section. First, we summarize the known results.

In Ref. [89] it is proven that the smallest eigenvalue of an uncorrelated Wishart matrix converges almost surely to the lower edge of the Marčenko-Pastur (4.141) distribution. Its fluctuations are observed to be similar to those of the largest eigenvalue as long as $p/n = \gamma^2 < 1$ [91]. As such they are of the order $\mathcal{O}(p^{1/3})$ [185]. These fluctuations are studied using the gap probability (2.43) which we evaluate at the lower edge of the spectrum on the scale of the fluctuations. Mathematically this is equivalent to appropriately centering and rescaling the eigenvalues,

$$\frac{x_i - \mu_-}{\sigma_-} \ , \tag{5.117}$$

where μ_- and σ_- are yet to be determined. Under modest assumptions on the distribution of the data matrix entires W_{ij}, the authors of Ref. [185] prove using the scaling

$$\mu_- = \left(\sqrt{p} - \sqrt{n}\right)^2 \ , \quad \sigma_- = \left(\sqrt{p} - \sqrt{n}\right)\left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{n}}\right)^{1/3} \ . \tag{5.118}$$

that the limiting distribution of the smallest eigenvalue of the resulting model correlation matrix is Tracy-Widom, whenever $n - p$, n and p tend to infinity and

$p/n = \gamma^2 < 1$ is fixed. Because $p \leq n$, the rescaling σ_- is negative which is not a problem. To our knowledge, no results exist in the literature about the limiting distribution of smallest eigenvalue for correlated Wishart matrices in the limit p/n fixed while n, p tending to infinity, even not in the complex case where the joint probability distribution function is known.

To begin with our analysis, we consider the eigenvalue integral representation (5.94) of the gap probability (2.43). The part of the integrand depending on the empirical eigenvalues of gap probabilities (5.94) and (5.105) are similar. Thus, we apply the analysis of the previous section to

$$E_p^{(\beta)}([0,s];0) = K \int \mathrm{d}[Y] \, \frac{|\Delta_p(Y)|^\beta \exp\left(\gamma_2 \mathrm{tr}\left(\imath Y - \epsilon \mathbf{1}_p\right)\right)}{\prod\limits_{k,i=1}^{p,p} \left(1 + 2\Lambda_k \left(\imath y_i - \epsilon\right)/(s\beta)\right)^{\beta/2}} \times \int \mathrm{d}[Q] \, \Theta(Q) \mathrm{det}^{(n-p)/\gamma_1}(Q + \mathbf{1}_{p\gamma_2}) \exp\left(\mathrm{tr}\left(\imath Y \otimes \mathbf{1}_{\gamma_2} - \epsilon \mathbf{1}_{\gamma_2 p}\right)\right) Q) \ . \tag{5.119}$$

As above, we set $s = \mu_-(\Lambda) + \sigma_-(\Lambda)\chi$ and assume that $\mu_-(\Lambda)$ and $\sigma_-(\Lambda)$ are both large. Moreover, we take the empirical eigenvalues to be of the order $\mathcal{O}(1)$ with variance

$$\mathrm{Var}_s(\Lambda) \sim \mathcal{O}(p^{-c'}) \ . \tag{5.120}$$

Employing the Tschebyscheff inequality, we approach the empirical eigenvalues as in Eq. (5.110) with c replaced by c'. Inserting this into Eq. (5.119) and expanding it with respect to large s, p and n we arrive at

$$E_p^{(\beta)}\left([0,s];0\right) = E_p^{(\beta)}\left([0,s];0\right)\Big|_{\Lambda=\bar{\Lambda}\mathbf{1}_p} - \frac{\mathrm{tr}\Lambda^{(1)}}{p^{c'} p \bar{\Lambda}} s \frac{\mathrm{d}}{\mathrm{d}s} E_p^{(\beta)}\left([0,s];0\right)\Big|_{\Lambda=\bar{\Lambda}\mathbf{1}_p} + \mathcal{O}(p^{-c'}) \ . \tag{5.121}$$

The leading order in p of the above expansion is the probability to find no eigenvalue below $[0, s]$ in the uncorrelated Wishart model with variance $\bar{\Lambda}$. If we center and rescale the threshold parameter as follows $s = \bar{\Lambda}\mu_- + \sigma_-\bar{\Lambda}\chi$, we find that the first term converges to

$$E_p^{(\beta)}\left([0,s];0\right) \to 1 - F_\beta(\chi) \ . \tag{5.122}$$

By the same argument as above, $\mathrm{tr}\Lambda^{(1)}/p$ converges to a constant such that it remains to analyze the derivative in the first and all higher correction terms. It leads to Eq. (5.114) with c replaced by c'. Thus, if $c' > 2/3$ all corrections terms in the expansion (5.121) tend to zero and $1 - E_p^{(\beta)}([0,s];0)$ converges to $F_\beta(\chi)$.

As for the largest eigenvalue $c' > 2/3$ ensures a macroscopic distance between the fluctuations of the smallest eigenvalue and that of the empirical eigenvalues such that the statistics decouple. We summarize our findings in the following statement.

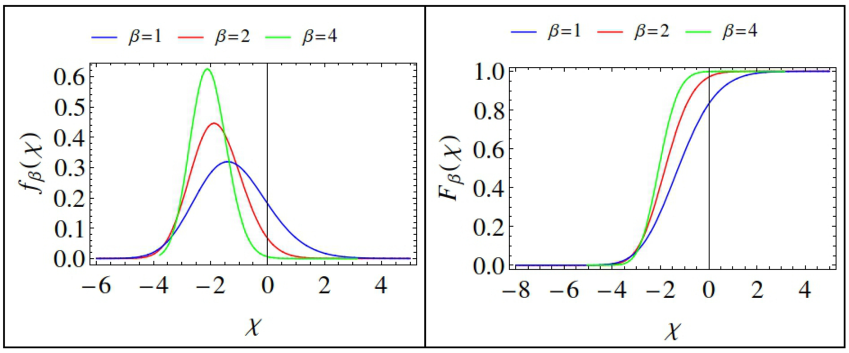


Figure 5.1: Left: The Tracy-Widom distribution $f_\beta(\chi) = \mathrm{d}F_\beta(\chi)/\mathrm{d}\chi$ for $\beta = 1, 2, 4$. Right : The cumulative density function $F_\beta(\chi)$ of the Tracy-Widom distribution for $\beta = 1, 2, 4$.

Statement *: Let Λ be the distinct eigenvalues of the empirical correlation matrix C. We choose them such that all eigenvalues are of order $\mathcal{O}(1)$ with variance of order $\mathcal{O}(p^{-c'})$ and $c' > 2/3$. If n, p tend to infinity while $p/n = \gamma^2 < 1$ is fixed, we obtain*

$$1 - E_p^{(\beta)}([0, \mu_-(\Lambda) + \sigma_-(\Lambda)\chi]; 0) \to F_\beta(\chi) \,, \tag{5.123}$$

where $\mu_-(\Lambda) = \mu_-\bar{\Lambda}$, $\sigma_-(\Lambda) = \sigma_-\bar{\Lambda}$ with μ_-, σ_- as in Eq. (5.118) and F_β is the integrated Tracy-Widom distribution.

5.4.3 Numerical Simulations

To confirm our findings for the limiting distribution of the largest as well as the smallest eigenvalue, we compare numerical simulations of the real, the complex and the real quaternion Wishart model to the Tracy-Widom distribution.

We first perform a numerical analysis of the closed-form expression Tracy and Widom obtained for the cumulative density function in Refs. [177,178,181]. As shown in Eq. (5.99), (5.102) and (5.103) for $\beta = 2, 1, 4$, respectively. To solve this nonlinear differential equation numerically we follow Ref. [186] and write Eq. (5.100) as system of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\begin{bmatrix} q(\chi) \\ q'(\chi) \\ I(\chi) \\ I'(\chi) \\ J(\chi) \end{bmatrix} = \begin{bmatrix} q'(\chi) \\ \chi q(\chi) + 2q^3(\chi) \\ I'(\chi) \\ q^2(\chi) \\ -q(\chi) \end{bmatrix} \,, \tag{5.124}$$

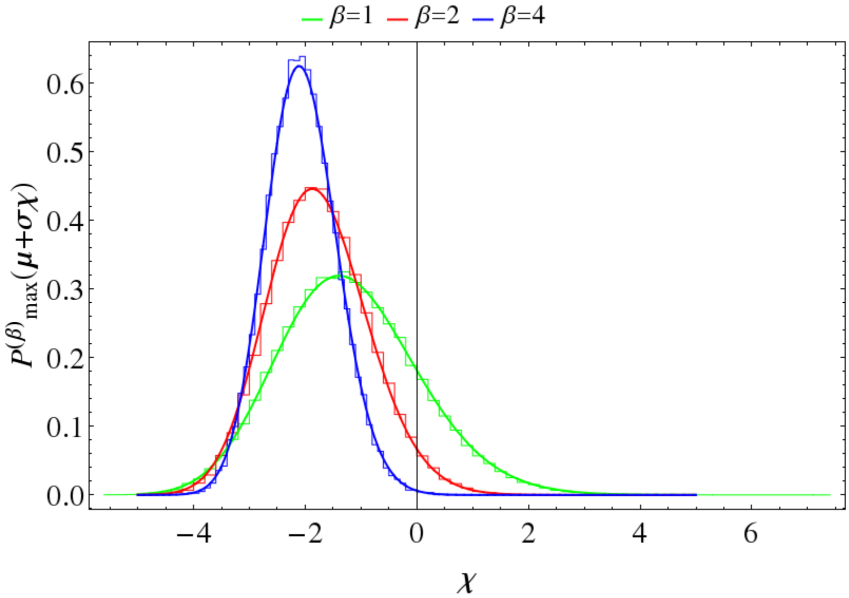


Figure 5.2: The Tracy-Widom distributions (lines) compared to numerical simulations (histograms) of the largest eigenvalue distribution. For the simulations we generate samples of 80 000 real, complex and real quaternion Wishart correlation matrices of size 100×300.

with the initial condition

$$\begin{bmatrix} q(\chi_0) \\ q'(\chi_0) \\ I(\chi_0) \\ I'(\chi_0) \\ J(\chi_0) \end{bmatrix} = \begin{bmatrix} Ai(\chi_0) \\ Ai'(\chi_0) \\ \int\limits_{\chi_0}^{\infty} (x-\chi_0) Ai^2(x) \mathrm{d}x \\ Ai^2(\chi_0) \\ \int\limits_{\chi_0}^{\infty} Ai(x) \mathrm{d}x \end{bmatrix} , \tag{5.125}$$

for χ_0 sufficient large. For clarity and comprehensibility, we introduce the quantities

$$I(\chi) = \int\limits_{\chi_0}^{\infty} (x-\chi) q^2(x) \mathrm{d}x \quad \text{and} \quad J(\chi) = \int\limits_{\chi_0}^{\infty} q(x) \mathrm{d}x \ . \tag{5.126}$$

This system of ordinary first order differential equations (5.124) can be solved numerically with standard software packages. We implement it into Mathematica [137] and show the numerical solution of Eq. (5.124), where $\chi_0 = 3.8$, in Fig. 5.1.

We start with the limiting largest eigenvalue statistics. To meet the requirements of the statement in section 5.4.1, we take the empirical eigenvalues to be uniformly distributed. The distribution is fixed if we take $\mathrm{Var}_s(\Lambda) = p^{-7/4}$ and $\langle \Lambda \rangle_s = 1 = \bar{\Lambda}$. We generate for each ensemble, the real, the complex and the real quaternion a sample of 80 000 Wishart model correlation matrices consisting of 100×300 dimensional data matrices. The latter have a mean correlation structure according to a set of empirical eigenvalues generated from the uniform distribution such that $n^{2/3}\mathrm{Var}_s(\Lambda) \approx 0.013 \ll 1$. In Fig. 5.2 we show the comparison between the numerical determined, centered and rescaled largest eigenvalue distribution and the analytic results for the Tracy-Widom distribution. For all three ensembles, we find perfect agreement between our analytic findings and the numerical simulations.

From the same ensemble, we computed the smallest eigenvalue distribution ensuring that we meet the requirements of the second statement in the previous section. A comparison between the numerical simulations and our analytic findings shows for all three ensembles a perfect agreement, see Fig. 5.3.

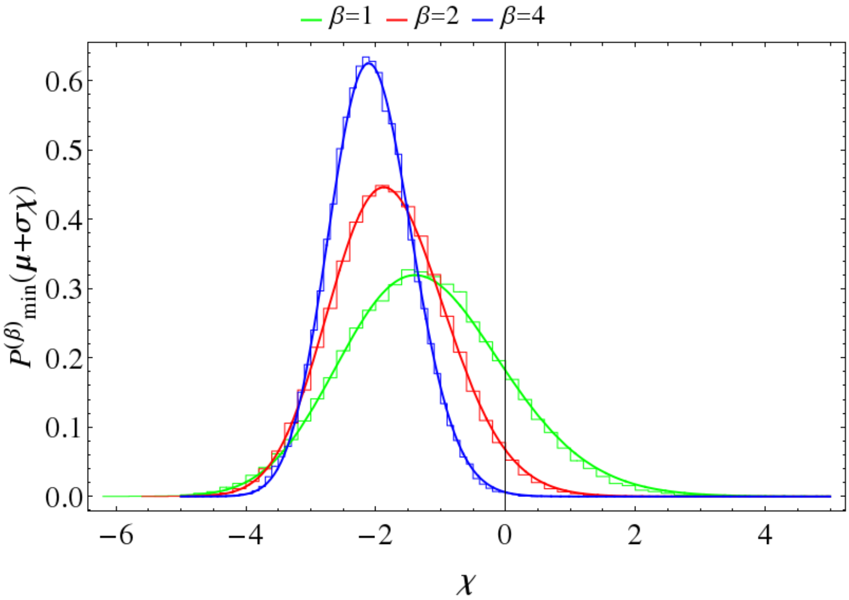


Figure 5.3: The Tracy-Widom distributions (lines) compared to numerical simulations (histograms) of the smallest eigenvalue distribution. For the simulations we generate samples of 80 000 real, complex and real quaternion Wishart model correlation matrices of size 100×300.

To demonstrate the agreement with the numerical simulations, we adjust the centering without changing the limit behavior. For the smallest eigenvalue, we adjust the scaling by a constant shift of the order $\mathcal{O}(1/n)$. This adjustment of the scaling is not necessary for the largest eigenvalue because it is not significantly affected by the 1/n correction.

5.5 Summary Chapter 5

In this chapter we developed a new approach based on a Fourier transform and exploiting the dyadic structure of the model correlation matrix $WW^\dagger$ to map non-invariant averages in the correlated Wishart model under modest assumptions to an invariant matrix model.

We employed it to construct for all three ensembles in a unified way a previously unknown eigenvalue integral representations for the gap probability related to the largest eigenvalue of a Wishart matrix. The expressions obtained for the complex and the real-quaternion Wishart model are analyzed using standard techniques from random matrix theory. For the complex case, we were able to recover the known results and derived a new determinantal expression in terms of known functions. For the real quaternion model, no closed-form expression exist in the literature. We obtained, for the first time, a Pfaffian point process for the gap probability. Remarkably, by this observation, we fully outmaneuvered the unitary-symplectic Itzykson-Zuber integral.

Because of square roots of characteristic polynomials arising in the most relevant real case, the standard methods of random matrix theory did not apply. Nonetheless, we were able to show that this quantity possesses a Pfaffian structure with a matrix kernel in terms of a coupled twofold integral. The latter depends on all empirical eigenvalues and appears to be hardly computable.

From these expressions, if the empirical eigenvalues are twofold degenerate, the square roots become integer powers. We exploited the asymptotic relation between two Wishart matrix models with and without a degeneracy in the empirical eigenvalues obtained in section 4.4 and assumed a twofold degeneracy. The eigenvalue integral for the gap probability obtained thereby is of standard kind such that random matrix techniques did apply. It led to a Pfaffian with a matrix kernel, where the entries are given in terms of transcendent functions depending on two empirical eigenvalues only. Although, this expression for the gap probability might be considered cumbersome, it led to unexpected insights. Importantly, the complexity of the resulting matrix kernel is not caused by the orthogonal Itzykson-Zuber integral, but is due to the observable we studied. On the other hand, there is no clue when considering the orthogonal Itzykson-Zuber integral (2.21) that one is able to derive a Pfaffian expression for the gap probability. Moreover, even with an assumed degeneracy in the empirical eigenvalues any summing up of an expression in terms of Jack polynomials and therefore summing up of the hypergeometric functions of matrix argument remains an unsurmountable task.

In the last part of this thesis, we analyzed the limiting statistics of the smallest and largest eigenvalue using the expressions for the gap probabilities derived by applying the Fourier approach. For a particular set of empirical eigenvalues, we were able to trace the limiting statistics of these quantities in the correlated Wishart model back to those in the uncorrelated Wishart model. The latter have been extensively studied in the literature, where it is proved that they are Tracy-Widom distributed. Importantly, this induces that the same is true for both quantities in the correlated Wishart model. We confirmed our findings with numerical simulations and found perfect agreement for both eigenvalues in all three ensembles.

CHAPTER 6

Conclusion and Outlook

Established as null hypothesis under the assumption of Gaussian statistics to quantify the empirically estimated correlation matrix in a sample of time series, the correlated Wishart model still challenges statisticians and physicists up until the present day. In this context the eigenvalues of the correlation matrix bear an outstanding significance. Their statistics, however, elude any exclusive analytical analysis even under the assumption of Gaussian statistics. Thus, only very special cases have been considered so far. On this account more or less all problems in the eigenvalue statistics of correlated Wishart matrices, except the level density, are still unsolved for more than three decades now.

In the meantime mathematicians and statisticians developed the theory of Jack polynomials, which facilitate the derivation of analytic expressions for statistical quantities depending only on the eigenvalues of the Wishart matrix. These have the drawback of being exclusively constructible and evaluable on a computer such that a further analytic consideration of the resulting expressions is hardly possible.

Remedy is provided by the Fourier approach I developed and the supersymmetry method which I extended to include also correlated Wishart matrices with an arbitrary distribution. In applications to the eigenvalue statistics it turned out that both approaches complement each other. While the former applied well to the statistics of the extreme eigenvalues, the latter was very convenient for studying the bulk eigenvalue statistics within the correlated Wishart model. The approaches have in common that both exploit the positive definiteness of the model correlation matrix.

In detail I focused on the statistics of the extreme eigenvalues and in addition considered aspects of the bulk statistics in the real, the complex and the real quaternion correlated Wishart model in a unified way. I uncovered unknown analytic structures like invariant matrix models for gap probabilities related to the extreme eigenvalue statistics, dualities between invariant and non-invariant matrix models, Pfaffian and determinantal expressions, an asymptotic relation between Wishart matrix models and obtained universalities in the eigenvalue statistics like the Tracy-Widom distribution.

In the uncorrelated real and real quaternion Wishart model the smallest eigenvalue statistics are still challenging because of square roots of characteristic polynomials. Similar challenges arise in many other considerations. Hence, the Pfaffian structures and the exact expression I derived in chapter 3 show that at least in this case it is possible to circumvent this difficulty and to apply random matrix theory results.

In the more involved correlated ensembles the absence of a closed-form expression for the joint eigenvalue distribution function constitutes the main challenge. Supersymmetry turned out to be one of the most promising approaches to face this difficulty. My analysis of the extreme eigenvalue statistics emphasized that not only large matrix size asymptotics are considerably easier but also the analysis of statistical quantities in finite dimensional matrix ensembles is possible. Moreover, with the duality between an infinite and a finite dimensional invariant matrix model for the cumulative density function of the largest eigenvalue and the derivation of the asymptotic relation between degenerate and non-degenerate real correlated Wishart models, the method of supersymmetry facilitated insights which had not been expected from any other approach.

The invariant matrix models I observed for the gap probabilities, related to the smallest and the largest eigenvalue statistics, raised the question of an underlying principle. A distinct feature of the correlated Wishart ensemble compared to other non-invariant matrix models is the positive definiteness of its constituents $WW^\dagger$. In the development of the Fourier approach I exploited this property and derived invariant matrix models for statistical quantities on a general level. The eigenvalue integrals I obtained thereby can be solved for the complex and the real quaternion models by applying standard results from random matrix theory; the eigenvalue integrals for the real ensembles turned out to be solvable in the special case of doubly degenerate empirical eigenvalue spectra. This observation suggested that we fully circumvented the Itzykson-Zuber integral, because the occurring difficulties were caused by the observable I considered and not by the Itzykson-Zuber integral.

The generality and variety of my calculations and results, respectively, concerning the eigenvalue statistics of correlated Wishart matrices establish new possibilities to extend and quantify the considered null hypothesis. It raises questions such as to what extend it is possible to model the empirically observed heavy tails and what are the consequences for the eigenvalue statistics. Moreover, my observations have the potential to gain insights into the structure of the Jack polynomials, which besides their appearance in random matrix theory occurred also in other areas of physics and mathematics and are therefore of common interest.

APPENDIX A

Supplemental Material Chapter 3

A.1 Construction of the Orthogonal Polynomials for the Real Wishart-Laguerre Ensemble

We give a detailed derivation of the skew-orthogonal polynomials introduced in section 3.1.2. To apply the method of anticommuting variables, we write Eq. (3.21) as an integral over a full matrix space. A Wishart model representing $R_n^{(\eta)}(y,s)$, is fixed by the form of the Jacobian coming from the diagonalization of the Wishart correlation matrix. We introduce a real rectangular $n \times (n+1+2i)$ matrix $\widehat{W}$. The volume element of such a matrix decomposes by diagonalization $\widehat{W}\widehat{W}^T \to OXO^T$ into

$$\mathrm{d}[\widehat{W}] \ \sim |\Delta_p(X)| \det{}^i X \mathrm{d}[X] \ \mathrm{d}\mu(O) \ , \tag{A.1}$$

where $O^T \in \mathrm{O}(n)$ and $\mathrm{d}\mu(O)$ is the Haar measure. Integrating over the orthogonal group yields a constant factor. A comparison of the volume element (A.1) and the eigenvalue integral (3.21) leads to

$$R_n^{(\eta)}(y,s) = K_n' \int \mathrm{d}[\widehat{W}] \ \det\left(\widehat{W}\widehat{W}^T - y\mathbf{1}_n\right) \frac{\exp\left(-\frac{\eta}{2}\mathrm{tr}\widehat{W}\widehat{W}^T\right)}{\sqrt{\det\left(\widehat{W}\widehat{W}^T + s\mathbf{1}_n\right)}} \ . \tag{A.2}$$

In section 2.5, we discussed application of supersymmetry to correlated Wishart ensembles. Applying these results for $\Lambda = \mathbf{1}_n$ to the present case we arrive at a supermatrix model of dimension 3×3 with an one dimensional bosonic block,

$$R_n^{(\eta)}(y,s) = K_n' \int \mathrm{d}[\mu] \ \ \mathrm{sdet}^{-n/2}\left(\frac{\eta}{2}T - \mu\right) \exp\left(-\mathrm{str}\mu\right) I_1(\mu) \ , \tag{A.3}$$

where

$$I_{n+2i+1}(\mu) = \int \mathrm{d}[\nu] \ \ \mathrm{sdet}^{-(n+2i+1)/2}\nu \exp\left(-\mathrm{str}\mu\nu\right) \tag{A.4}$$

is the supersymmetric Ingham-Siegel integral [113]. The domain of integration is the set of Hermitian $(1|2)\times(1|2)$ supermatrices, with a self dual fermionic block and a positive real bosonic block. The supermatrix μ is parametrized by

$$\mu = \left[\begin{array}{ccc} x & \zeta^* & \zeta \\ \zeta & \imath b & 0 \\ -\zeta^* & 0 & \imath b \end{array}\right] , \tag{A.5}$$

where ζ is a complex Grassmann and x and b are real variables. The flat measure on the superspace reads

$$\mathrm{d}[\mu] \ = \mathrm{d}b\mathrm{d}x\mathrm{d}\zeta^*\mathrm{d}\zeta \ . \tag{A.6}$$

We use a similar parametrization and integration measure for ν in Eq. (A.4) and introduce the matrix T in Eq. (A.3) which is a $(1|2) \times (1|2)$-dimensional diagonal supermatrix given by

$$T = \left[\begin{array}{ccc} -s & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & y \end{array}\right] . \tag{A.7}$$

In principle, we can expand the full expression (A.3) in Grassmann variables and perform the remaining integrations. Instead of doing so, we obtain a form of Eq. (A.2) which consists of two ordinary integrals only and is related to the results of Edelman [129].

A representation without Grassmannians occurs, because the matrix model (A.2) is invariant under the action $\widehat{W} \to O_L\widehat{W}O_R$, where $O_L \in \mathrm{O}(n)$ and $O_R \in \mathrm{O}(n+1+2i)$ are independent. Utilizing this symmetry and replacing in the determinant in the denominator the invariant $\widehat{W}\widehat{W}^T$ by $\widehat{W}^T\widehat{W}$ yields

$$\begin{aligned} R_n^{(\eta)}(y,s) = K_n'\sqrt{s}^{2i+1} \int \mathrm{d}[\widehat{W}]\ & \frac{\det\left(\widehat{W}\widehat{W}^T - y\mathbf{1}_n\right)}{\sqrt{\det\left(\widehat{W}^T\widehat{W} + s\mathbf{1}_{n+1+2i}\right)}} \\ \times \exp\left(-\frac{\eta}{2}\mathrm{tr}\widehat{W}\widehat{W}^T\right) & \ . \end{aligned} \tag{A.8}$$

The advantage is that we can apply the results of section 2.5 separately for the determinant in the numerator and denominator such that we will finally obtain an ordinary matrix rather than supermatrix integral.

To construct this matrix model, we write the determinant in the numerator and the determinant to half-integer power in the denominator as Gaussian integrals over a n-dimensional complex Grassmannian and a $(n+1+2i)$-dimensional real vector. We order the Grassmannian vector in a $n \times 2$ matrix and its Hermitian conjugate,

$$A = \left[\begin{array}{cc} \xi_1 & \xi_1^* \\ \vdots & \vdots \\ \xi_n & \xi_n^* \end{array}\right] , \quad A^\dagger = \left[\begin{array}{ccc} \xi_1^* & \dots & \xi_n^* \\ -\xi_1 & \dots & -\xi_n \end{array}\right] , \tag{A.9}$$

and perfrom the $\widehat{W}$ integral, yielding

$$
\begin{aligned}
R_n^{(\eta)}(y,s) = K_n'\sqrt{s}^{2i+1} \int \mathrm{d}[A]\ \mathrm{d}[v]\ \exp\left(-\frac{s}{2}v^Tv - \frac{y}{2}\mathrm{tr}A^\dagger A\right) \\
\times \det{}^{-1/2}\left(\left(\eta \mathbf{1}_{n+1+2i} + vv^T\right)\otimes \mathbf{1}_n + \mathbf{1}_{n+1+2i}\otimes AA^\dagger\right)\ ,
\end{aligned}
\tag{A.10}
$$

where $\mathrm{d}[A] = \prod_i^n \mathrm{d}\xi_i \mathrm{d}\xi_i^*$. Hence, we see the advantage of the replacement of $WW^\dagger$ by $W^\dagger W$ in Eq. (A.8). It decouples the parts due to the determinant in the numerator from those due to the determinant in the denominator. We exploit an one-to-one correspondence between the invariants of $AA^\dagger$, $A^\dagger A$ and v^Tv, vv^T [109, 113], see Eq. (2.110), and rewrite the integrand (A.10) as

$$
\begin{aligned}
R_n^{(\eta)}(y,s) = K_n'\sqrt{s}^{2i+1} \int \mathrm{d}[A]\ \mathrm{d}[v]\ \det{}^{1/2}\left(\left(\eta + v^Tv\right)\mathbf{1}_2 + A^\dagger A\right) \\
\times \frac{\det^{n/2+i}\left(\eta \mathbf{1}_2 + A^\dagger A\right)}{(\eta + v^Tv)^{n/2+1}} \exp\left(-\frac{s}{2}v^Tv - \frac{y}{2}\mathrm{tr}A^\dagger A\right)\ .
\end{aligned}
\tag{A.11}
$$

Since it depends on the 2×2 matrix $A^\dagger A$ and the scalar v^Tv only, we reformulate the quantities in terms of a matrix integral within the appropriate symmetry class and of an integral over the positive real line, respectively. To this end, we use bosonization [156] and find

$$
\begin{aligned}
R_n^{(\eta)}(y,s) = K_n'\sqrt{s}^{2i+1} \int \mathrm{d}\mu(V) \int_0^\infty \mathrm{d}x x^{(n+2i-1)/2} \frac{\det^{n/2+i}\left(\eta \mathbf{1}_2 + V\right)}{(\eta + x)^{n/2+1}} \\
\times \det{}^{-n/2} V \det{}^{1/2}\left(\left(\eta + x\right)\mathbf{1}_2 + V\right) \exp\left(-\frac{s}{2}x - \frac{y}{2}\mathrm{tr}V\right)\ ,
\end{aligned}
\tag{A.12}
$$

where $V \in \mathrm{CSE}(2) = \mathrm{U}(2)/\mathrm{USp}(2)$ and $\mathrm{d}\mu(V)$ is the corresponding Haar measure. Because of the symplectic structure, we have $V = \mathrm{diag}\,(z,z)$ such that the n-dimensional eigenvalue integral (3.21) of the orthogonal polynomial is reduced to a twofold integral,

$$
\begin{aligned}
R_n^{(\eta)}(y,s) = K_n'\sqrt{s}^{2i+1} \oint \frac{\mathrm{d}z}{z^{n+1}} \left(\eta + z\right)^{n+2i} \exp\left(-yz\right) \\
\times \int_0^\infty \mathrm{d}x \frac{x^{(n-1+2i)/2}\left(\eta + x + z\right)}{(\eta + x)^{n/2+1}} \exp\left(-\frac{s}{2}x\right)\ ,
\end{aligned}
\tag{A.13}
$$

where the z-contour encloses zero. Both integrals can be considered separately, because they are coupled by a linear term only. The remaining integrals are of the form [136]

$$
\begin{aligned}
\oint \frac{\mathrm{d}z}{z^{n+1}} \left(\eta + z\right)^{n+m} \exp\left(-yz\right) &= \frac{2\pi\imath\eta^m(-1)^n}{n!} \sum_{i=0}^{n} \frac{(n+m)!n!(-1)^{n+i}}{(n-i)!(m+i)!} \frac{(\eta y)^i}{i!} \\
&= \frac{2\pi\imath\eta^m(-1)^n}{n!} L_n^{(m)}(\eta y)\ ,
\end{aligned}
\tag{A.14}
$$

where $L_n^{(m)}$ are the monic Laguerre polynomials, *i.e.* $L_n^{(m)}(x) = x^n + \dots$ and

$$\int_0^\infty \mathrm{d}x \frac{x^{(n-m')/2}}{(\eta+x)^{n/2+1}} \exp\left(-\frac{s}{2}x\right) = \frac{\Gamma\left(\frac{n-m'+2}{2}\right)}{\eta^{m'/2}} \mathrm{U}\left(\frac{n-m'+2}{2}, 1-\frac{m'}{2}; \frac{\eta s}{2}\right) \ , \tag{A.15}$$

where $\mathrm{U}(a,b;z)$ is known as the confluent hypergeometric function [136]. Inserting everything into Eq. (A.13), we arrive at the announced formula for the polynomials

$$\begin{aligned} R_n^{(\eta)}(y,s) =& K_n' \sqrt{s}^{2i+1} \left[\mathrm{U}\left(\frac{n+2i+1}{2}, \frac{1+2i}{2}; \frac{\eta s}{2}\right) L_n^{(1+2i)}(\eta y) \right. \\ & \left. + \frac{n+2i+1}{2} \mathrm{U}\left(\frac{n+3+2i}{2}, \frac{3+2i}{2}; \frac{\eta s}{2}\right) L_n^{(2i)}(\eta y) \right] \ . \end{aligned} \tag{A.16}$$

For later purpose we compute the normalization constant K_n' by comparing Eq. (A.16) with the definition of the polynomial (3.21) in the limit $\lim_{s\to\infty} s^{-n/2} R_n^{(\eta)}(-s,s)$. This yields

$$K_n' = \frac{(-1)^n 2^{(n-2i-1)/2}}{\eta^{(n(n+2i+2)-2i-1)/2}} \prod_{j=0}^{n-1} (j+1)! \prod_{k=1}^{i} \frac{(n-1+2k)!}{(2k)!} \ . \tag{A.17}$$

Collecting everything, we derive the desired orthogonal polynomials with respect to the weight (3.7) using Eq. (3.19) and (3.20). For even degree they read

$$\begin{aligned} \frac{R_{2N}^a(y,s)}{K_{2N} s^{(2i+1)/2}} =& \frac{2N!}{(2N-a)!} \left[\mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{1+2i}{2}; \frac{s}{2}\right) L_{2N-a}^{(1+2i+a)}(y) \right. \\ & \left. + \frac{2N+2i+1}{2} \mathrm{U}\left(\frac{2N+2i+3}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) L_{2N-a}^{(2i+a)}(y) \right] \ , \end{aligned} \tag{A.18}$$

and

$$\begin{aligned} & \frac{R_{2N+1}^a(y,s)}{K_{2N+1} s^{(2i+1)/2}} = \left(y + 2N(2N+2i+2) - 2i - 1\right) R_{2N}^a(y,s) \\ & + a R_{2N}^{a-1}(y,s) - \frac{(2N)!}{(2N-a)!} \left(\mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{1+2i}{2}; \frac{s}{2}\right) \right. \\ & \times \left[2y(2N-a) L_{2N-a-1}^{(2+a+2i)}(y) + 2a L_{2N-a}^{(1+a+2i)}(y) \right] \\ & + \frac{2N+2i+1}{2} \mathrm{U}\left(\frac{2N+2i+3}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) \\ & \times \left[s L_{2N-a}^{(2i+1+a)}(y) - 2y(2N-a) L_{2N-a-1}^{(a+1+2i)}(y) - 2a L_{2N-a}^{(a+2i)}(y) \right] \\ & \left. + s \frac{2N+2i+1}{2} \frac{2N+2i+3}{2} \mathrm{U}\left(\frac{2N+2i+5}{2}, \frac{5+2i}{2}; \frac{s}{2}\right) L_{2N-a}^{(2i+a)}(y) \right) \end{aligned} \tag{A.19}$$

for odd degree, where we absorb all constants into the new normalization constants K_{2N} and K_{2N+1} and set $L_N^{(b)}(y) = 0$ for all $N < 0$. The superscript a in Eq. (A.18) and (A.19) denotes the ath derivative with respect to the "polynomial" argument y, which we computed for later purpose.

We are left with calculating the normalization constants K_i and the scalar products $r_j(s)$. The former are determined from Eq. (A.18) and Eq. (A.19) using the normalization condition (3.22). Employing it leads to

$$K_{2N} = K_{2N+1} = \frac{1}{\sqrt{s}^{2i+1} \mathrm{U}\left(\frac{2N+2i+1}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)} \ . \tag{A.20}$$

The more involved task is to compute the scalar products, because we have to insert the polynomials (A.18) and (A.19) into Eq. (3.16) and to perform the integrals. Although the resulting integrals involve Laguerre polynomials only, we choose an alternative derivation. We make use of the following identity [40]

$$Z_{2m}^{0,w_i}(0) = \int\limits_0^\infty \mathrm{d}x_1 \ldots \int\limits_0^\infty \mathrm{d}x_{2m} \, |\Delta_{2m}(X)| \prod_{j=1}^{2m} w_i(x_j,t) = (2m)! \prod_{j=0}^{m-1} r_j(s) \tag{A.21}$$

and find

$$r_m(s) = \frac{(2m)!}{(2m+2)!} \frac{Z_{2m+2}^{0,w_i}(0)}{Z_{2m}^{0,w_i}(0)} \ . \tag{A.22}$$

For the partition functions emerging in the above expression, we already know analytic formulas, but we still have to evaluate proportionality constants. We write

$$Z_N^{0,w_i}(0) = C_{N,N+1+2i}^{-1} \int \mathrm{d}[B] \, \frac{\exp\left(-\frac{1}{2}\mathrm{tr} BB^\dagger\right)}{\sqrt{\det\left(BB^\dagger + s\mathbf{1}_N\right)}} \tag{A.23}$$

where B is a real $\mathbb{R}^{N\times(N+1+2i)}$ matrix and $C_{N,N+1+2i}$ is the volume of $\mathrm{O}(N) \times \mathrm{O}(N+2i+1)$. As the volume is independent from the function we are integrating, we can determine the constant $C_{N,N+2i+1}$ using the Laguerre weight function only, yielding

$$\begin{aligned} C_{p,n} &= \frac{\int\limits_{\mathbb{R}^{p\times n}} \mathrm{d}[F] \, \exp\left(-\frac{1}{2}\mathrm{tr} FF^\dagger\right)}{\int\limits_{[0,\infty)^p} \mathrm{d}[X] \, |\Delta_p(X)| \prod_{i=0}^{p} x_i^{(n-p-1)/2} \exp\left(-\frac{x_i}{2}\right)} \\ &= \frac{\sqrt{2\pi}^{pn} \, \Gamma^p(3/2)}{2^{pn/2} \prod_{j=0}^{p-1} \Gamma\left(\frac{j+3}{2}\right) \Gamma\left(\frac{j+n-p+1}{2}\right)} \ . \end{aligned} \tag{A.24}$$

The integral in the denominator is a Selberg integral [40]. The full matrix integral shown in Eq. (A.23) is proportional to a limit of the orthogonal polynomial discussed above. From Eq. (A.2) we obtain

$$\begin{aligned}\int \mathrm{d}[B]\, \frac{\exp\left(-\frac{1}{2}\mathrm{tr}BB^\dagger\right)}{\sqrt{\det\left(BB^\dagger + s\mathbf{1}_N\right)}} &= \lim_{y\to\infty} \frac{(-1)^p}{y^p} R_N^{(1)}(y,s) \\ =& \frac{\sqrt{2\pi}^{N(N+2i+1)}\sqrt{s}^{2i+1}}{2^{(N+2i+1)/2}} \mathsf{U}\left(\frac{N+2i+1}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) .\end{aligned} \tag{A.25}$$

Combining everything we find for the partition function (A.23)

$$\begin{aligned}Z_{2m}^{0,w_i}(0) = 2^{(2m-2i-1)/2} \prod_{j=0}^{2m-1} (j+1)! \prod_{k=0}^{i-1} \frac{(2m+1+2k)!}{(2k+1)!} \\ \times \sqrt{s}^{2i+1} \mathsf{U}\left(\frac{2m+2i+1}{2}, \frac{3+2i}{2}; \frac{s}{2}\right) .\end{aligned} \tag{A.26}$$

If we use Eq. (A.22) and insert expression (A.26) into it, the normalizations constants of the scalar products $r_m(s)$ read

$$r_m(s) = 2(2m)!(2m+1+2i)! \frac{\mathsf{U}\left(\frac{2m+3+2i}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)}{\mathsf{U}\left(\frac{2m+1+2i}{2}, \frac{3+2i}{2}; \frac{s}{2}\right)} , \tag{A.27}$$

for $m = 0, \ldots, l-1$.

A.2 Asymptotic Analysis of Orthogonal Polynomials for the Gap Probability

We derive asymptotic expressions for the skew-orthogonal polynomials (3.23) and (3.24) with $i = 0$ used to compute closed-form expressions for the gap probability in section 3.1.3. They consist of Laguerre polynomials and confluent hypergeometric functions. For these we know the asymptotic behavior,

$$\mathsf{U}\left(ap+c, b; \frac{u}{8p}\right) = \frac{p^{b-1}2}{\Gamma\left(ap+c\right)} \sqrt{\frac{a8}{u}}^{b-1} \mathsf{K}_{b-1}\left(2\sqrt{\frac{ua}{8}}\right) + \mathcal{O}(p^{b-2}), \tag{A.28}$$

where K_b is the modified Bessel function of second kind and

$$\frac{(-1)^{ap+c} L_{ap+c}^{(b)}\left(\frac{-u}{4p}\right)}{\Gamma\left(ap+c+1\right)} = p^b \sqrt{\frac{a4}{u}}^b \mathsf{I}_b\left(\sqrt{ua}\right) + \mathcal{O}(p^{b-1}) , \tag{A.29}$$

where I_b is the modified Bessel function of first kind. Both expressions are derived using their integral representations [136]. Substituting Eq. (A.28) and (A.29) into the orthogonal polynomial with even degree (A.18) we find

$$\begin{aligned}R_{x(p+2m-2)}^a\left(\frac{-u}{4p}, \frac{u}{4p}\right) = (-2)^a \Gamma\left(x(p+2m-2)+1\right) p^a \\ \sqrt{\frac{x}{u}}^a \left(\mathsf{I}_{a+1}\left(\sqrt{ux}\right) + \mathsf{I}_a\left(\sqrt{ux}\right)\right)\end{aligned} \tag{A.30}$$

According to the skew-symmetry of the kernel, see Eq. (3.14), the term in Eq. (A.19) proportional to R^a_{2N} drops out such that only the remaining part of R^a_{2N+1} is important for the asymptotic considerations. If we utilize

$$\frac{\mathsf{K}_{3/2}(z)}{\mathsf{K}_{1/2}(z)} = 1 + \frac{1}{z} \;, \tag{A.31}$$

we obtain that the leading order term of R^a_{2N+1} in p important for the skew-symmetric product is

$$\begin{aligned}&\frac{(-2)^a}{2}\Gamma\left(x(p+2m-2)+1\right)\\ &\times\sqrt{\frac{xp^2}{u}}^{a}\left[\left(2-4a-\sqrt{ux}\right)\mathsf{I}_a\left(\sqrt{ux}\right)+\left(4-\sqrt{ux}\right)\mathsf{I}_{a+1}\left(\sqrt{ux}\right)\right] \;.\end{aligned} \tag{A.32}$$

The large p behavior of the $r_j(s)$ are derived readily and are given by

$$r_{x(p+2m-2)/2}\left(\frac{u}{4p}\right) \approx 4\Gamma^2\left(x(p+2m-2)+1\right) \;. \tag{A.33}$$

Since the r_j only occur in the denominator of the Pfaffian kernel (3.14), the Γ-functions coming from the asymptotic analysis of the polynomials cancels with those in Eq. (A.40).

A.3 Asymptotic Analysis of the Normalization Constant for the Gap Probability

An asymptotic analysis of the prefactor in Eq. (3.32) can be done using Eq. (A.28) and

$$\Gamma\left(ap+c\right) \approx \sqrt{2\pi p}(ap)^{ap+c}\exp\left(-ap\right) \;. \tag{A.34}$$

We perform the asymptotic analysis of the prefactor stepwise. We first study the part of $\mathcal{C}_{p,n}$ constant when varying s and find, employing Eq. (A.34),

$$\begin{aligned}\mathcal{C}_{p,n} \approx &\frac{2^{(2m-1)/2}\prod_{i=0}^{\alpha-1}(2i)!}{\sqrt{\pi}\prod_{i=0}^{\alpha-1}i!}\Gamma\left(\frac{p+2m+1}{2}\right)\\ &\times\exp(-(\alpha-2m)p)\sqrt{2\pi p}^{(2m-\alpha)}p^{(2m-\alpha)p+2m(2m-1)/2-\alpha(\alpha-1)+m-2\alpha}\end{aligned} \tag{A.35}$$

It is noteworthy that only the leading order in p of Eq. (3.33) and Eq. (A.35) are equal, sub-leading terms differ. Clearly, if we insert $\alpha = 2m$ or $\alpha = 2m-1$ into the expression above it simplifies drastically. The second step in the analysis of the prefactor relies on the asymptotic behavior of the confluent hypergeometric function for $s = u/4p$,

$$\exp\left(-ps/2\right)\sqrt{s}\,\mathsf{U}\left(\frac{p+2m+1}{2},\frac{3}{2};\frac{s}{2}\right) \approx \frac{\exp\left(-u/8\right)}{\Gamma\left(\frac{p+2m+1}{2}\right)}\left(\frac{u}{4}\right)^{1/4}\mathsf{K}_{1/2}\left(\sqrt{u/4}\right) \;. \tag{A.36}$$

A.4 Asymptotic Analysis of Orthogonal Polynomials for the Smallest Eigenvalue Distribution

Asymptotic expressions for the polynomials and the kernel are derived in a similar fashion as in section 3.1.3. For the polynomials of even degree and the important part of the polynomials with odd degree, we find

$$\begin{aligned}R^a_{x(p+m-3)}(-\frac{u}{4p},\frac{u}{4p}) = &\frac{(-1)^b 2^{b+2} p^{b+2} \left(x/u\right)^{(b+2)/2}}{2/\sqrt{ux}+1}\Gamma\left(2mx+px-3x+1\right)\\ &\times\left(\left(\frac{2}{\sqrt{ux}}+1\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right)+\mathsf{I}_{b+3}\left(\sqrt{ux}\right)\right)\end{aligned} \tag{A.37}$$

and

$$\begin{aligned}&\frac{(-1)^{b+1}2^{b+1}p^{b+2}\left(\frac{x}{u}\right)^{(b+2)/2}}{\frac{2}{\sqrt{ux}}+1}\Gamma\left(px+2xm-3x+1\right)\\ &\times\left(\left(\sqrt{ux}-10\right)\mathsf{I}_{b+3}\left(\sqrt{ux}\right)+\left(2(2b-3)\left(\frac{2}{\sqrt{ux}}+1\right)+\sqrt{ux}\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right)\right)\ ,\end{aligned} \tag{A.38}$$

respectively. We approximate for large values of p the sums in Eq. (3.43) and (3.45) by integrals as introduced in Eq. (3.37). By inserting the asymptotic expression for the orthogonal polynomials into the kernel (3.14), its leading order in p is given by

$$\begin{aligned}&p^{a+b+3}\Xi^{(a,b)}(u) = (-1)^{a+b+1}p^{a+b+3}2^{a+b}\int\limits_0^1 \mathrm{d}x\frac{\left(\frac{x}{u}\right)^{(b+a+4)/2}}{\left(\frac{2}{\sqrt{ux}}+1\right)}x^2\\ &\times\Bigg[4\left(a-b\right)\left(\frac{2}{\sqrt{ux}}+1\right)\mathsf{I}_{a+2}\left(\sqrt{ux}\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right)\\ &+\left(\left(\sqrt{ux}-10\right)-\frac{ux}{2+\sqrt{ux}}-2\left(2b-3\right)\right)\mathsf{I}_{a+3}\left(\sqrt{ux}\right)\mathsf{I}_{b+2}\left(\sqrt{ux}\right)\\ &-\left(\left(\sqrt{ux}-10\right)-\frac{ux}{2+\sqrt{ux}}-2\left(2a-3\right)\right)\mathsf{I}_{b+3}\left(\sqrt{ux}\right)\mathsf{I}_{a+2}\left(\sqrt{ux}\right)\Bigg],\end{aligned} \tag{A.39}$$

where we used that

$$r_{x(p+2m-3)/2}\left(\frac{u}{4p}\right) = 4p^2x^2\Gamma^2\left(2mx+xp-3x+1\right)\ . \tag{A.40}$$

We do not perform a large p expansion of the Γ-function in Eq. (A.40), because it is canceled in any case.

A.5 Asymptotic Analysis of the Normalization Constant for the Smallest Eigenvalue Distribution

We compute the large-p behavior of the normalization constant (3.44). If we make use of the asymptotic expression for the Γ-function (A.34), it turns out that

$$\begin{aligned}\tilde{\mathcal{C}}_{p,n} \approx & \frac{(-1)^{\alpha(p-1)}2^{m-1}\prod_{j=0}^{\alpha-1}(2i)!}{\sqrt{\pi}\prod_{j=0}^{\alpha-1}i!}\Gamma\left(\frac{p+3}{2}\right) \\ & \times e^{-(2m-\alpha)p}(2\pi p)^{(2m-\alpha)/2}p^{(2m-\alpha)p+2m(2m-1)/2+(2m-\alpha)-\alpha(\alpha-1)} \, .\end{aligned} \tag{A.41}$$

To check that the leading order in p of $\tilde{\mathcal{C}}_{p,n}$ is finite for $p \to \infty$, we recall that we rescale not only $s \to u/4p$, but also $\mathcal{P}^{(1)}_{\text{min}}(s) \to \mathcal{P}^{(1)}_{\text{min}}(u/4p)/4p$. The factor $1/4p$ is caused by the fact that $\mathcal{P}^{(1)}_{\text{min}}(s)$ is a distribution. Therefore, rescaling of its argument results in

$$P^{(1)}_{\text{min}}(s)\ \text{d}s \mapsto P^{(1)}_{\text{min}}\left(\frac{u}{4p}\right)\ \text{d}\left(\frac{u}{4p}\right) \ , \tag{A.42}$$

for $s \mapsto u/4p$.

A.6 Pfaffian Expression in the Alternative Approach

The alternative approach is based on a general treatment of eigenvalue integrals in Ref. [112], for other approaches see Refs. [40,44,107,110] and references therein. We sketch the important intermediate steps only. The authors showed that the partition function (3.50) can be written in terms of a Pfaffian determinant with a $(M+\chi+4m)\times(M+\chi+4m)$-dimensional kernel

$$\begin{aligned}Z_M^{\alpha,w_i}(\kappa) = & \frac{(-1)^{M(M-1)/2+2mM}M!}{\Delta_{2m}(\kappa)} \\ & \times \text{pf}\left[\begin{array}{c|cc} 0_{2m\times 2m} & 0_{2m\times 1} & \kappa_b^{i-1} \\ \hline 0_{1\times 2m} & 0 & \int\limits_0^\infty \text{d}x x^{i-1}w(x;t) \\ -\kappa_a^{j-1} & -\int\limits_0^\infty \text{d}x x^{j-1}w(x;t) & \left\langle x^{j-1}\middle|x^{i-1}\right\rangle_{t,1}\end{array}\right] ,\end{aligned} \tag{A.43}$$

where $1 \leq i,j \leq M+2m$ and $1 \leq a,b \leq 2m$. If M is even, *i.e.* $\chi = 0$, the first row and column of the lower right block are omitted. We set $d = M+2m$ and denote the $(d+\chi)\times(d+\chi)$ dimensional skew-symmetric matrix, in the lower right block of the Pfaffian (A.43) by $B_{d+\chi}(s)$. It is the matrix of moments with respect to the weight $w_i(x;s)$.

To decrease the dimension of the Pfaffian kernel, we apply the Schur complement,

$$\text{pf}\left[\begin{array}{cc} A & B \\ -B^T & C\end{array}\right] = \text{pf}C\ \text{pf}\left[A+BC^{-1}B^T\right] , \tag{A.44}$$

which holds only if D is an invertible even dimensional matrix. Employing Schur complement (A.44), we decompose the Pfaffian of the $(M+\chi+4m)\times(M+\chi+4m)$ dimensional matrix in Eq. (A.43) into a product of two Pfaffians

$$\begin{aligned} Z_M^{2m,w_i}(\kappa) &= \frac{(-1)^{M(M-1)/2}M!}{\Delta_\alpha(\kappa)}\mathrm{pf}\,B_{d+\chi}(s) \\ &\times \mathrm{pf}\left[\sum_{i,j=1}^{d}\kappa_a^{i-1}\left(B_{d+\chi}^{-1}(s)\right)_{i+\chi,j+\chi}\kappa_b^{j-1}\right] . \end{aligned} \tag{A.45}$$

The Pfaffian of the matrix $B_{d+\chi}(s)$ is proportional to the partition function $Z_d^{0,w_i}(0)$. It can readily be derived and is given by

$$\mathrm{pf}\,B_{d+\chi}(s) = \frac{(-1)^{(M+2m)(M+2m-1)/2}}{(M+2m)!}Z_d^{0,w_i}(0) , \tag{A.46}$$

where $Z_d^{0,w_i}(0)$ was computed in section 3.1.2, see Eq. (A.26). If we use the invariance of the Pfaffian and choose a basis in the space of polynomials in κ_i such that

$$\kappa_a^{j-1} \mapsto R_{j-1}(\kappa_a), \tag{A.47}$$

where $R_{j-1}(\kappa_a)$ satisfy Eq. (3.15), $B_{d+\chi}(s)$ becomes diagonal and Eq. (A.45) becomes Eq. (3.13).

Alternatively, we consider Eq. (3.50) for $M=N$ and $\alpha=2$, where $N=2\tilde{L}+\tilde{\chi}$ is an arbitrary integer. The solution of this eigenvalue integral can readily be derived from Eq. (A.45)

$$Z_N^{2,w_i}(\kappa) = \frac{(-1)N!}{(\kappa_a-\kappa_b)(N+2)!Z_{N+2}^{0,w_i}(0)}\sum_{i,j=1}^{N+2}\kappa_a^{i-1}\left(B_{N+2+\tilde{\chi}}^{-1}(s)\right)_{i+\tilde{\chi},j+\tilde{\chi}}\kappa_b^{j-1} . \tag{A.48}$$

We find an expression similar to the matrix kernel in the Pfaffian determinant of Eq. (A.45). Thus, setting $N=d-2$ and substituting Eq. (A.48) into the Pfaffian (A.45), leads to

$$\begin{aligned} Z_M^{2m,w_i}(\kappa) &= \frac{(-1)^{2m(2m-1)/2}M!}{(M+2m)!\Delta_{2m}(\kappa)}Z_d^{0,w}(0) \\ &\times \mathrm{pf}\left[\frac{(-1)(\kappa_a-\kappa_b)(M+2m)!Z_{M+2m-2}^{2,w_i}(\kappa_a,\kappa_b)}{(M+2m-2)!Z_{M+2m}^{0,w_i}(0)}\right] . \end{aligned} \tag{A.49}$$

A.7 Constructing of a Dual Model in the Alternative Approach

We construct an ordinary, non-invariant matrix model dual to the eigenvalue integral (3.52). Analogous to the construction of the skew-orthogonal polynomials in appendix A.1, we first construct a Wishart matrix model representation of Eq. (3.52).

The Wishart model is fixed by the condition that diagonalization should lead to $Z^{2,w_i}_{d-2}(\kappa)$. The discussion below Eq. (3.22) yields

$$Z^{2,w_i}_{d-2}(\kappa) = K \int\limits_{\mathbb{R}^{d-2\times(d+2i-1)}} \mathrm{d}[\widehat{W}] \, \frac{\exp\left(-\frac{1}{2}\mathrm{tr}\widehat{W}\widehat{W}^\dagger\right)}{\sqrt{\det\left(\widehat{W}\widehat{W}^\dagger + s\mathbf{1}_{d-2}\right)}} \times \det\left(\widehat{W}\widehat{W}^\dagger - \kappa_a\mathbf{1}_{d-2}\right)\det\left(\widehat{W}\widehat{W}^\dagger - \kappa_b\mathbf{1}_{d-2}\right) \,, \tag{A.50}$$

where $\widehat{W}$ is a real $(d-2)\times(d+2i-1)$-dimensional matrix. In the denominator, we use the identity $\det\left(\widehat{W}\widehat{W}^\dagger + s\mathbf{1}_{d-2}\right) = \sqrt{s}^{2i+1}\det\left(\widehat{W}^\dagger\widehat{W} + s\mathbf{1}_{d+2i-1}\right)$. This will facilitate a decoupling of the boson-boson and the fermion-fermion blocks. We write the determinants in the numerator and the determinant in the denominator as Gaussian integrals over two $d-2$ dimensional complex vectors with Grassmannian entries and a $d+2i-1$ dimensional real vector. After algebraic manipulations, we perform the W integral and arrive at

$$Z^{2,w_i}_{d-2}(\kappa) = K\sqrt{s}^{2i+1}\int \mathrm{d}[A,v]\ \exp\left(-\frac{sv^Tv}{2} - \frac{1}{2}\mathrm{tr}\kappa A^\dagger A\right) \times \det{}^{-1/2}\left((vv^T + \mathbf{1}_{d+2i-1})\otimes\mathbf{1}_{d-2} + \mathbf{1}_{d+2i-1}\otimes AA^\dagger\right) \,, \tag{A.51}$$

where $\kappa = \mathrm{diag}(\kappa_a,\kappa_a,\kappa_b,\kappa_b)$. Because of the dyadic structure of $AA^\dagger$ and vv^T the invariants of these objects are in one-to-one correspondence with invariants of $A^\dagger A$ and v^Tv, respectively. If we employ this duality to the present case, we arrive at

$$Z^{2,w_i}_{d-2}(\kappa) = K\sqrt{s}^{2i+1}\int \mathrm{d}[A,v]\ \exp\left(-\frac{sv^Tv}{2} - \frac{1}{2}\mathrm{tr}\kappa A^\dagger A\right) \det{}^{1/2}\left(\left(v^Tv+1\right)\mathbf{1}_4 + A^\dagger A\right)\frac{\det^{(d+2i-2)/2}\left(\mathbf{1}_4 + A^\dagger A\right)}{(1+v^Tv)^{d/2+1}} \,. \tag{A.52}$$

Accordingly, the matrix model (A.52) only depends on the scalar v^Tv and the four dimensional matrix $A^\dagger A$. As we did in section 3.1.2, we replace these quantities by a scalar and a matrices integral using bosonization [156] such that Eq. (A.52) becomes

$$Z^{2,w_i}_{d-2}(\kappa) = K\sqrt{s}^{2i+1}\int\limits_0^\infty \frac{\mathrm{d}x x^{(d+2i-3)/2}}{(1+x)^{d/2+1}}\int \frac{\mathrm{d}\mu(V)\ \det^{(d+2i-2)/2}\left(\mathbf{1}_4 + V\right)}{\det^{(d-2)/2}V} \det{}^{1/2}\left((x+1)\,\mathbf{1}_4 + V\right)\exp\left(-\frac{sx}{2} - \frac{1}{2}\mathrm{tr}\kappa V\right) \,, \tag{A.53}$$

where $V\in\mathrm{CSE}(4) = \mathrm{U}(4)/\mathrm{USp}(4)$ and $\mathrm{d}\mu(V)$ is the corresponding Haar measure.

A.8 Finite n, p Expressions Using the Alternative Approach

We use the alternative approach to further simplify our finite and large M expressions. To this end we diagonalize the CSE matrix $V = U(r\otimes\mathbf{1}_2)U^\dagger$, where

$U \in \mathrm{USp}(4)$ and $r = \mathrm{diag}(r_1, r_2)$ with $r_i \in U(1)$ is the diagonal matrix of distinct eigenvalues. This coordinate change leads to a decomposition of the volume form

$$\mathrm{d}\mu(V) \sim (r_1 - r_2)^4 \det{}^{-3} r \mathrm{d}[r] \; \mathrm{d}\mu(U), \tag{A.54}$$

where $\mathrm{d}\mu(U)$ is the Haar measure on USp(4). Because of the symmetry breaking term in the exponential in Eq. (3.53) the resulting group integral becomes the unitary-symplectic Itzykson-Zuber integral. For the special case of $\mathrm{USp}(4)$, analytic expressions are known [67]

$$\begin{aligned} &\int \mathrm{d}\mu(U) \exp\left(-\frac{1}{2}\mathrm{tr}\kappa U(\mathbf{1}_2 \otimes r)U^\dagger\right) \\ \sim&\frac{\exp(-(r_1\kappa_a + r_2\kappa_b)/2) - \exp(-(r_1\kappa_b + r_2\kappa_a)/2)}{(r_1 - r_2)^3(\kappa_a - \kappa_b)^3/4} \\ +&\frac{\exp(-(r_1\kappa_a + r_2\kappa_b)/2) + \exp(-(r_1\kappa_b + r_2\kappa_a)/2)}{(r_1 - r_2)^2(\kappa_a - \kappa_b)^2} \; , \end{aligned} \tag{A.55}$$

where $\kappa = \mathrm{diag}(\kappa_a, \kappa_a, \kappa_b, \kappa_b)$. Substituting expression (A.55) into the kernel (3.53) and utilizing its invariance under the permutation of r_1, r_2 we obtain

$$\begin{aligned} Z_{d-2}^{2,w_i}(\kappa) = &\frac{K\sqrt{s}^{2i+1}}{(\kappa_a - \kappa_b)^3} \int\limits_0^\infty \frac{\mathrm{d}x x^{(d+2i-3)/2}}{(1+x)^{d/2+1}} \oint \frac{\mathrm{d}[r] \;\; (r_1 - r_2)}{\det^{d+1} r} \\ &\det{}^{d+2i-2}\left(\mathbf{1}_2 + r\right) \; \det\left((x+1)\,\mathbf{1}_2 + r\right) \exp\left(-\frac{sx}{2}\right) \\ &\left(1 + \frac{(\kappa_a - \kappa_b)}{2}(\partial_{\kappa_a} - \partial_{\kappa_b})\right) \exp(-(r_1\kappa_a + r_2\kappa_b)/2) \; , \end{aligned} \tag{A.56}$$

where $\partial_x = \partial/\partial x$. The remaining r-integral reduces to one double integral, if we use derivatives to express the determinant coupling the x and the r-integration by

$$\begin{aligned} &\det\left((x+1)\,\mathbf{1}_2 + r\right) \exp(-(r_1\kappa_a + r_2\kappa_b)/2) \\ =&(x+1)^2 + 2(x+1)\left(\partial_{\kappa_a} + \partial_{\kappa_b}\right) + 4\partial_{\kappa_a}\partial_{\kappa_b}) \exp(-(r_1\kappa_a + r_2\kappa_b)/2) \; . \end{aligned} \tag{A.57}$$

We insert this into Eq. (A.56) and take all derivatives out of the integral such that we are left with the following two-fold integral

$$\begin{aligned} &\oint \frac{\mathrm{d}[r] \;\; (r_1 - r_2)}{\det^{d+1} r} \det{}^{d+2i-2}\left(\mathbf{1}_2 + r\right) \exp(-(r_1\kappa_a + r_2\kappa_b)/2) \\ =&\frac{(2\pi\imath)^2}{d!(d-1)!}\left(L_{d-1}^{(2i-1)}(\kappa_a/2) L_d^{(2i-2)}(\kappa_b/2) - L_d^{(2i-2)}(\kappa_a/2) L_{d-1}^{(2i-1)}(\kappa_b/2)\right) \\ =&\frac{(2\pi\imath)^2 2}{d!^2}\left(\partial_{\kappa_a} - \partial_{\kappa_a}\right) L_d^{(2i-2)}(\kappa_a/2) L_d^{(2i-2)}(\kappa_b/2) \; . \end{aligned} \tag{A.58}$$

The x-integral leads to confluent hypergeometric functions, see Eq. (A.15) such that the two point kernel is given by

$$
\begin{aligned}
&Z_{d-2}^{2,w_i}(\kappa) = \frac{K\sqrt{s}^{2i+1}}{(\kappa_a-\kappa_b)^3}\left(1+\frac{(\kappa_a-\kappa_b)}{2}(\partial_{\kappa_a}-\partial_{\kappa_b})\right)(\partial_{\kappa_a}-\partial_{\kappa_a}) \\
&\left[\mathrm{U}\left(\frac{d+2i-1}{2},\frac{2i+3}{2};\frac{s}{2}\right)+4\mathrm{U}\left(\frac{d+2i-1}{2},\frac{2i-1}{2};\frac{s}{2}\right)\partial_{\kappa_a}\partial_{\kappa_b}\right. \\
&\left.+\,2\mathrm{U}\left(\frac{d+2i-1}{2},\frac{2i+1}{2};\frac{s}{2}\right)(\partial_{\kappa_a}+\partial_{\kappa_b})\right]L_d^{(2i-2)}(\kappa_a/2)L_d^{(2i-2)}(\kappa_b/2)\ .
\end{aligned}
\tag{A.59}
$$

The normalization constant is determined by comparing Eq. (3.50) for $M=d-2$, $\alpha=2$ and Eq. (A.59) for particular values of κ_a and κ_b. We rescale both expression by $(\kappa_a\kappa_b)^{d-2}$ and consider $\kappa_a,\kappa_b\to\infty$, yielding

$$
K = \frac{2^{(d-2i+2)/2+2(d-2)}}{d(d-1)}\prod_{j=0}^{d-3}(j+1)!\prod_{k=0}^{i-1}\frac{(d+2i-1)!}{(2k+1)!} \tag{A.60}
$$

A.9 Constructing a Pfaffian Structure for Eq. (3.64)

This section is devoted to a detailed construction of the Pfaffian expression (3.70). We follow the ideas of Ref. [112] and introduce p integrals over δ-functions such that we arrive at

$$
\begin{aligned}
Z_p^{\nu/0}(\kappa) = &\int_0^\infty \mathrm{d}x_1\ldots\mathrm{d}x_{2p}\prod_{i<j}(x_i-x_j)\prod_{i=1}^{p}g(x_i,x_{i+p};s) \\
&\times\prod_{a=1}^{\nu}(x_i-\kappa_a)(x_{i+p}-\kappa_a)\ ,
\end{aligned}
\tag{A.61}
$$

where $g(x,z;s)=w(x;s)\delta(x-z)(x-z)^{-1}$. Combining the Vandermonde determinant with the product of the κ's to a Vandermonde determinant of a $2p+\nu$ dimensional matrix, leads to

$$
\prod_{i<j}(x_i-x_j)\prod_{i,a=1}^{2p,\nu}(x_i-\kappa_a) = \frac{\Delta_{2p+\nu}(x_1,\ldots,x_{2p},\kappa_1,\ldots,\kappa_\nu)}{\Delta_\nu(\kappa_1,\ldots,\kappa_\nu)} \tag{A.62}
$$

$$
= \frac{(-1)^{(2p+\nu)(2p+\nu-1)/2}}{\Delta_\nu(\kappa_1,\ldots,\kappa_\nu)}\det\left[x_i^{j-1}\ \ x_{i+p}^{j-1}\ \ \kappa_a^{j-1}\right]\ , \tag{A.63}
$$

where $1\le j\le 2p+\nu$, $1\le a\le\nu$ and $1\le i\le p$. Because of the base independence of the determinant, we add up rows or columns without changing the results. Thus we rotate the basis such that a monomial x_j^{j-1} is replaced by a monic polynomial of degree $j-1$

$$
x_i^{j-1}\mapsto Q_{j-1}(x_i) = x_i^{j-1}+\ldots \tag{A.64}
$$

If we substitute this into the ν-point partition function (A.61), the resulting integral is covered by de Bruijn's integral theorem, *c.f.* Ref. [111]. Employing it, we find

$$Z_p^{\nu/0}(\kappa) = \frac{(-1)^{(2p+\nu)(2p+\nu-1)/2+p(p-1)/2+\nu(\nu-1)/2}p!}{\Delta_\nu(\kappa_1,\ldots,\kappa_\nu)} \\ \text{pf}\left[\begin{array}{cc} 0 & Q_{j-1}(\kappa_b) \\ -Q_{i-1}(\kappa_a) & \int\limits_0^\infty dx w(x;s)\left(Q_{i-1}(x)Q'_{j-1}(x) - Q_{i-1}(x)Q'_{j-1}(x)\right) \end{array}\right] , \tag{A.65}$$

where i, j and a, b are as above. The entries of the $(2p+\nu)\times(2p+\nu)$ dimensional lower right block consist of the a skew-symmetric scalar product with respect to the weight $w(x;s)$,

$$\langle f, g\rangle_{4,s} = \int\limits_0^\infty dx w(x;s)\left(f(x)g'(x) - f'(x)g(x)\right) . \tag{A.66}$$

Yet we did not say anything about the choice of polynomials $Q_j(x)$. We choose them to be skew-orthogonal with respect to the scalar product (A.66),

$$\langle Q_{2j+1}|Q_{2i}\rangle_{4,s} = r_i(s)\delta_{ij} , \quad \langle Q_{2j+1}|Q_{2i+1}\rangle_{4,s} = \langle Q_{2j}|Q_{2i}\rangle_{4,s} = 0 , \tag{A.67}$$

such that the lower right block in the Pfaffian (A.65) becomes block-diagonal. The resulting two-by-two blocks have $r_i(s)$ and $-r_i(s)$ on the anti-diagonal. If $\nu = 2m$ is even $2p+\nu$ is even as well and we can apply the Schur complement (A.44) to Eq. (A.65), leading to

$$Z_p^{\nu/0}(\kappa) = \frac{(-1)^{(2p+2m)(2p+2m-1)/2+p(p-1)/2+2m(2m-1)/2}p!}{\Delta_{2m}(\kappa_1,\ldots,\kappa_{2m})}\text{pf}M_{2p+2m}(s) \\ \times \text{pf}\left[\sum_{j=0}^{p+m-1}\frac{Q_{2j+1}(\kappa_a)Q_{2j}(\kappa_b) - Q_{2j}(\kappa_a)Q_{2j+1}(\kappa_b)}{r_j(s)}\right] , \tag{A.68}$$

where $1 \leq a, b \leq 2m$ and $M_{2p+2m}(s)$ is the $(2p+2m)\times(2p+2m)$ dimensional block in the lower right of the Pfaffian (A.65). The Pfaffian of $M_{2p+2m}(s)$ is related to $Z_{p+m}^{0/0}(0)$. To see this, we repeat the analysis from Eq. (A.61) to Eq. (A.65) for $\nu = 0$ and read off

$$\text{pf}M_{2p+2m}(s) = \frac{(-1)^{(2p+2m)(2p+2m-1)/2+(p+m)(p+m-1)/2}}{(p+m)!}Z_{p+m}^{0/0}(0) . \tag{A.69}$$

where the partition function on the right hand side of Eq. (3.71) is a Selberg integral of the form of Eq. (3.66) with $\nu = 0$.

A.10 Orthogonal Polynomials for the Real Quaternion Wishart Model

We present a detailed derivation of the three-by-three matrix model (3.88). The Pfaffian determinants in Eq. (3.86) are expressed as Gaussian integrals over anti-commuting variables, see Eq. (2.67), yielding

$$\begin{aligned} &\mathrm{pf}^2\left[\begin{array}{cc} B^\dagger & \sqrt{y}\mathbf{1}_{2n} \\ -\sqrt{y}\mathbf{1}_{2n} & -B \end{array}\right]\mathrm{pf}\left[\begin{array}{cc} B^\dagger & \sqrt{t}\mathbf{1}_{2n} \\ -\sqrt{t}\mathbf{1}_{2n} & B \end{array}\right] \\ \sim &\int d[\zeta]\ \exp\left(-2\mathrm{tr}\left(\begin{array}{cc} 0 & -B \\ B^\dagger & 0 \end{array}\right)\Omega_{2n}\zeta\zeta^T + 2\mathrm{tr}\lambda\zeta^T\Omega_{2n}\zeta\right)\ , \end{aligned} \tag{A.70}$$

where Ω_{2n} is as introduced in Eq. (2.108). We introduce $\zeta = [(\zeta_{i1}),(\zeta_{i2}),(\zeta_{i3})]$ and

$$\lambda = \left(\begin{array}{ccc} \imath\sqrt{y} & 0 & 0 \\ 0 & \imath\sqrt{y} & 0 \\ 0 & 0 & \sqrt{t} \end{array}\right)\ . \tag{A.71}$$

Substituting this integral representation into the matrix model for the generating polynomial (3.86), exchanging the ζ and the B integrals, we arrive at

$$\begin{aligned} Q_{2n}^{(\eta)}(y) = K\int d[\zeta]\ \exp\left(2\mathrm{tr}\lambda\zeta^T\Omega\zeta\right) \\ \times\int d[B]\ \exp\left(-\eta 2\mathrm{tr}\left(\begin{array}{cc} 0 & -B \\ B^\dagger & 0 \end{array}\right)^2 - 2\mathrm{tr}\left(\begin{array}{cc} 0 & -B \\ B^\dagger & 0 \end{array}\right)\Omega_{2n}\zeta\zeta^T\right)\ . \end{aligned} \tag{A.72}$$

As such, the B-integral reduces to a standard Gaussian with a source term. It is most convenient to separate it into an integral over its real and imaginary part. Performing the resulting $(2n-1)(2n-2)$ real Gaussian integrals leads to

$$Q_{2n}^{(\eta)}(y) = K\int d[\zeta]\ \exp\left(2\mathrm{tr}\lambda\zeta^T\Omega_{2n}\zeta - \frac{1}{4\eta^2}\mathrm{tr}\zeta^T\Omega_{2n}\zeta\zeta^T\Omega_{2n}\zeta\right)\ . \tag{A.73}$$

The matrix $\zeta^T\zeta$ is a three dimensional, anti-symmetric matrix with real, commuting entries. In a product including Ω_{2n} as occurring in Eq. (A.73), $\zeta^T\Omega_{2n}\zeta$ is a real symmetric matrix. Thus, to replace it by an ordinary matrix, we can apply the generalized Hubbard-Stratonovich transformation or bosonization. Employing bosonization [156], the resulting average is given by

$$Q_{2n}^{(\eta)}(y) = K\int d[O]\ \det{}^{-2n-1}O\exp\left(2\mathrm{tr}\lambda O - \frac{1}{4\eta^2}\mathrm{tr}O^2\right)\ , \tag{A.74}$$

where $O \in \mathrm{COE}(3) = \mathrm{U}(3)/\mathrm{O}(3)$.

APPENDIX B

Supplemental Material Chapter 4

B.1 Derivation of Known Results

We give a detailed derivation of the exact expression for the gap probability in the complex correlated Wishart discussed in section 4.1.3. To solve the eigenvalue integral (4.22), we need to know a closed-form expression for $f_{p,2}$. To this end, we repeat a calculation done in Ref. [109]. $f_{p,2}$ is given by the Fourier transform of a determinant to a positive integer power and has to be interpreted as distribution on the space of Hermitian matrices. To compute it, we diagonalize the Hermitian matrix ρ in Eq. (4.15) and find

$$f_{p,2}(\hat{s}) \sim \int\limits_{-\infty}^{\infty} \mathrm{d}[r] \;\; \Delta_v^2(r) \; \mathrm{det}^p r \; \int \mathrm{d}\mu(u) \; \exp\left(-\imath \mathrm{tr} \; uru^\dagger \hat{s}\right) \; , \tag{B.1}$$

where $\mathrm{d}\mu(u)$ is the Haar measure on $\mathrm{U}(v)$. The group integral is the unitary Harish-Chandra-Itzykson-Zuber integral. It can be solved analytically and is given by [95, 96]

$$\int \mathrm{d}\mu(u) \; \exp\left(-\imath \mathrm{tr} uru^\dagger \hat{s}\right) \sim \frac{\det\left[\exp\left(-\imath r_i \hat{s}_j\right)\right]}{\Delta_v(r)\Delta_v(\hat{s})} \; . \tag{B.2}$$

The remaining r-integrand (B.1) is invariant under permutations of the r_i. Hence, when substituting the right hand side of Eq. (B.2) into it, the determinant in the numerator of Eq. (B.2) reduces to one term and we are left with

$$f_{p,2}(\hat{s}) \sim \frac{1}{\Delta_v(\hat{s})} \int\limits_{-\infty}^{\infty} \mathrm{d}[r] \;\; \Delta_v(r) \;\; \prod_{i=1}^{v} r_i^p \exp\left(-\imath r_i \hat{s}_i\right) \tag{B.3}$$

$$\sim \frac{1}{\Delta_v(\hat{s})} \det\left[\frac{\partial^{p+j-1}}{\partial \hat{s}_i^{p+j-1}} \delta(\hat{s}_i)\right] \; , \tag{B.4}$$

where $i, j = 1, \ldots, v$. From the first to the second line in the expression above, we use $\Delta_v(r) = (-1)^{v(v-1)/2} \det r_i^{j-1}$ and take the integrals into the determinant.

We get rid of the determinant structure in Eq. (B.4) if we use properties of the δ-function, leading to an expression factorizing in the eigenvalues of σ

$$f_{p,2}(\hat{s}) \sim \frac{1}{\Delta_v(\hat{s})}\det\left[\hat{s}_i^{v-j}\frac{\partial^{p+v-1}}{\partial\hat{s}_i^{p+v-1}}\delta(\hat{s}_i)\right] \sim \prod_{i=1}^{v}\frac{\partial^{p+v-1}}{\partial\hat{s}_i^{p+v-1}}\delta(\hat{s}_i)\ . \tag{B.5}$$

An alternative derivation of this result is given in Ref. [113]. The authors used a differential operator for the derivation. The expression obtained will be useful when we consider the gap probability for the real correlated Wishart ensemble.

If we replace $f_{p,2}$ in Eq. (4.22) by Eq. (B.5), we obtain an eigenvalue integral with a fully factorizing weight function

$$\begin{aligned} E_p^{(2)}([0,s];0) = K\exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right)\int \mathrm{d}[\hat{s}]\ \ |\Delta_v(\hat{s})|^2\exp\left(\mathrm{tr}\hat{s}\right) \\ \times\prod_{k=1,i=1}^{p,v}\left(s+\Lambda_k\hat{s}_i\right)\prod_{i=1}^{v}\frac{\partial^{p+v-1}}{\partial\hat{s}_i^{p+v-1}}\delta(\hat{s}_i)\ . \end{aligned} \tag{B.6}$$

As consequence, we can take all factors including the integral into the determinant. This reduces the computation of the v integrals to a calculation of one integral in the determinant kernel,

$$\begin{aligned} E_p^{(2)}([0,s];0) &= K\exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right) \\ \times\det&\left[\int\limits_{-\infty}^{\infty}\mathrm{d}z z^{i+j-2}\exp\left(z\right)\prod_{k=1}^{p}\left(s+\Lambda_k z\right)\frac{\partial^{p+v-1}}{\partial z^{p+v-1}}\delta(z)\right]\ , \end{aligned} \tag{B.7}$$

where $1 \leq i,j \leq v$. The p-fold product can be written as polynomial in z with elementary symmetric polynomials $e_k(\Lambda)$ as coefficients

$$\prod_{k=0}^{p}\left(s+\Lambda_k z\right) = \sum_{k=0}^{p} s^{p-k}e_k(\Lambda)z^k\ , \tag{B.8}$$

where e_k denotes the kth elementary symmetric function. It reads

$$e_k(\Lambda) = \sum_{1\leq i_1<\cdots<i_k\leq p}\Lambda_{i_1}\cdots\Lambda_{i_k}\ . \tag{B.9}$$

For example, the first three elementary symmetric functions are

$$e_0(\Lambda) \equiv 1\ , \tag{B.10}$$
$$e_1(\Lambda) = \Lambda_1+\Lambda_2+\cdots+\Lambda_p\ , \tag{B.11}$$
$$e_2(\Lambda) = \Lambda_1\Lambda_2+\Lambda_1\Lambda_3+\cdots+\Lambda_{p-1}\Lambda_p\ . \tag{B.12}$$

If we insert the series expansion (B.8) into the integral in Eq. (B.7), the resulting integration is simple. We first move by partial integration all derivatives from the δ-function to the remaining integrand. All boundary terms caused by partial integration are zero such that the integral is up to an overall sign the $p+v-1$-fold derivative of the remaining integrand evaluated at zero,

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} \mathrm{d}z z^{i+j-2} \exp{(z)} \prod_{k=1}^{p} (s+\Lambda_k z) \frac{\partial^{p+v-1}}{\partial z^{p+v-1}} \delta(z) \\ =&(-1)^{p+v-1}(p+v-1)!\Theta(\alpha_2) \sum_{k=0}^{\min(p,\alpha_2)} \frac{s^{p-k} e_k(\Lambda)}{(\alpha_2-k)!} \ ,\end{aligned} \tag{B.13}$$

where $\alpha_2 = p+v+1-i-j$ and $1 \leq i,j \leq v$. Substituting the matrix kernel (B.13) into the determinantal expression for the gap probability (B.7) leads then to a closed form expression for the gap probability

$$\begin{aligned}E_p^{(2)}([0,s];0) =& \frac{(-1)^{v(v-1)/2}}{\det^v \Lambda} \exp\left(-\mathrm{tr}\frac{s}{\Lambda}\right) \\ \times \det &\left[\Theta(\alpha_2) \sum_{k=0}^{\min(p,\alpha_2)} \frac{s^{p-k} e_k(\Lambda)}{(\alpha_2-k)!}\right] \ .\end{aligned} \tag{B.14}$$

where we use the $E_p^{(2)}([0,s];0) \to 1$ for $s \to 0$ to compute the overall normalization constant K.

B.2 Derivation of Eq. (4.28)

This section is devoted to a detailed derivation of the exact expression for the gap probability in the real correlated Wishart discussed in section 4.1.3. According to the factorization of $f_{p+1,1}$ in terms of the eigenvalues $\hat{s}_i$, see Eq. (4.27), we apply a standard result from random matrix theory and express the integral over the v eigenvalues (4.26) together with Vandermonde determinant as a Pfaffian determinant of a particular matrix kernel [40]

$$\begin{aligned}&E_p^{(1)}([0,s];0) = K \exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right) \\ \times&\mathrm{pf}\left[(j-i)\int\limits_{-\infty}^{\infty} \mathrm{d}z z^{i+j-3} \exp{(2z)} \prod_{k=0}^{p} \left(\frac{s}{2}+\Lambda_k z\right) \partial_{\hat{z}}^{p+2v-1}\delta(z)\right] \ ,\end{aligned} \tag{B.15}$$

where $1 \leq i,j \leq 2v$. We perform the integral in the kernel of the Pfaffian determinant in a similar fashion as in the case of the complex ensemble. We first expand the p-fold product using the identity (B.8) and then evaluate the remaining integral using partial integration. This leads to

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} \mathrm{d}z z^{i+j-3} \exp{(2z)} \prod_{k=0}^{p} \left(\frac{s}{2}+\Lambda_k z\right) \partial_z^{p+2v-1}\delta(z) \\ =&2^{2v-i-j+3}(-1)^{p-1}(p+2v-1)!\Theta(\alpha_1) \sum_{k=0}^{\min(p,\alpha_1)} \frac{e_k(\Lambda)}{(\alpha_1-k)!} s^{p-k} \ ,\end{aligned} \tag{B.16}$$

where $\alpha_1 = p+2v-i-j+2$. If we insert this into Eq. (B.15) and use the properties of the Pfaffian determinant, we arrive at

$$E_p^{(1)}([0,s];0) = \frac{\exp\left(-\mathrm{tr}\frac{s}{2\Lambda}\right)}{\det^v \Lambda}\mathrm{pf}\left[\sum_{k=0}^{\min(p,\alpha_1)} \frac{(j-i)\Theta(\alpha_1)\ e_k(\Lambda)}{(\alpha_1-k)!} s^{p-k}\right] , \tag{B.17}$$

where $1 \leq i,j \leq 2v$. The normalization constant was determined using $E_p^{(1)} \to 1$ for $s \to 0$.

B.3 Microscopic Limit of the Gap Probability

In the eigenvalue integral (4.47), except of the Vandermonde determinant, the whole integrand factorizes in the eigenvalues $\hat{s}_i$. Therefore it is possible to apply standard results of random matrix theory [40, 44] leading to a Pfaffian or a determinantal expression for $\beta = 1,2$, respectively. Applying these results to Eq. (4.47), we find

$$\mathcal{E}^{(\beta)}(u) = K\exp\left(-\frac{\beta u}{8}\right)\det{}^{\beta/2}\left[q_{ij}\oint \mathrm{d}z z^{i+j+m} e^{\sqrt{u}/2\left(z+z^{-1}\right)}\right] , \tag{B.18}$$

where $1 \leq i,j \leq 2v/\beta$, $m = -4+\beta-2v/\beta$ and $q_{ij} = (j-i)$ for $\beta = 1$ and $q_{ij} = (-1)^{i-1}$ for $\beta = 2$. For $\beta = 1$, the determinant to half-integer power is a Pfaffian determinant, because in this case the matrix kernel is skew-symmetric and by construction even dimensional. We reduced the computation of the v eigenvalue integrals to the calculation of a single integral in the matrix kernel. To simplify this contour integral, we make use of the following identity [136]

$$\exp\left(\sqrt{u}/2\left(z+z^{-1}\right)\right) = \sum_{k=-\infty}^{\infty} z^k \mathrm{I}_k(\sqrt{u}) \ ,\forall z \neq 0 \ , \tag{B.19}$$

where $\mathrm{I}_l(z)$ is the Bessel function of second kind. Substituting this identity into the matrix kernel, exchanging the integration and the summation, we obtain that the resulting integral projects out one Bessel function such that

$$\oint \mathrm{d}z z^{i+j+m} e^{\sqrt{u}/2\left(z+z^{-1}\right)} = 2\pi\imath\left(\frac{\beta\sqrt{u}}{4}\right)^{i+j+m+1} \mathrm{I}_{i+j+m+1}\left(\sqrt{u}\right) . \tag{B.20}$$

If we insert this into the gap probability (B.18), we arrive at

$$\mathcal{E}^{(\beta)}(u) = \exp\left(-\frac{\beta u}{8}\right)\det{}^{\beta/2}\left[q_{ij}\left(\frac{\beta\sqrt{u}}{2}\right)^{i+j+m+1} \mathrm{I}_{i+j+m+1}\left(\sqrt{u}\right)\right] , \tag{B.21}$$

where $m = -4+\beta-2v/\beta$, $q_{ij} = (j-i)$ for $\beta = 1$ and $q_{ij} = (-1)^{i-1}$ for $\beta = 2$ and $1 \leq i,j \leq 2v/\beta$. We determine the normalization constant using the expansion of the modified Bessel function of second kind for $z \ll 1$ [136]

$$\mathrm{I}_l(z) \sim \frac{(z/2)^{|l|}}{\Gamma(|l|+1)} , \quad \text{for } l = 0, \pm 1, \pm 2, \ldots \tag{B.22}$$

and the requirement that $\mathcal{E}^{(\beta)}(u) \to 1$ for $u \to 0$.

B.4 Empirical Eigenvalue Dependence of Eq. (4.87)

To see this on the level of the F, B averages (4.87), we perform the following change of coordinates $F \to \sqrt{C_F}F$ $B \to \sqrt{C_F}B$ and obtain

$$Z_p^{k/k}(\kappa) = K \int \mathrm{d}[F,B] \, \frac{\prod_{a=1}^{k} \det\left(\mathcal{H} - \kappa_{a2}\mathbf{1}_{\gamma_2 p}\right)}{\prod_{b=1}^{k} \det\left(\mathcal{H} - \kappa_{b1}\mathbf{1}_{\gamma_2 p}\right)} P(F|\mathbf{1}_{\gamma_2 p}) P(B|C_{\text{eff}}) \;, \tag{B.23}$$

where $\sqrt{C_F}$ is a square matrix with the property $\sqrt{C_F}\sqrt{C_F}^{\dagger} = C_F$. We make use of the fact that $\mathcal{H}$ does not change if we rescale B and F in the same way. This is because the contributions from the numerator and the denominator in Eq. (4.86) cancel each other. Since the resulting B and F integral is invariant under base changes, the generating function depends only on the eigenvalues of $C_{\text{eff}} = U\hat{\Lambda}_{\text{eff}}U^{\dagger}$, where $U \in \mathrm{G}_p$ and $\hat{\Lambda}_{\text{eff}} = \mathbf{1}_{\gamma_2} \otimes \Lambda_{\text{eff}}$ with $\Lambda_{\text{eff}} = \mathrm{diag}(\Lambda_1, \ldots, \Lambda_p)$.

B.5 Construction of the Two-Supermatrix Model

In this section we apply the analysis of section 2.5 to the two-Wishart matrix model (4.89) and derive a two-supermatrix model. To begin with, we write the determinants in the denominator and numerator of Eq. (4.89) as Gaussian integrals over vectors with complex ordinary and Grassmannian entries such that

$$\prod_{a=1}^{k} \frac{\det\left(FF^{\dagger}\frac{1-\kappa_{a,2}}{1+\kappa_{a,2}} - BB^{\dagger}\right)}{\det\left(FF^{\dagger}\frac{1-\kappa_{a,1}}{1+\kappa_{a,1}} - BB^{\dagger}\right)} = \int \mathrm{d}[A] \; \exp\left(\imath \mathrm{tr} FF^{\dagger}A\mathbf{j}A^{\dagger} + \imath \mathrm{tr} BB^{\dagger}AA^{\dagger}\right) \;. \tag{B.24}$$

In the expression (B.24) we make use of the rectangular supermatrix A introduced in Eq. (2.81), (2.76) and (2.82) and define

$$\mathbf{j} = \mathbf{1}_{\tilde{\gamma}} \otimes \mathrm{diag}\left(\frac{1-\kappa_{1,1}}{1+\kappa_{1,1}}, \ldots, \frac{1-\kappa_{k,1}}{1+\kappa_{k,1}}, \frac{1-\kappa_{1,2}}{1+\kappa_{1,2}}, \ldots, \frac{1-\kappa_{k,2}}{1+\kappa_{k,2}}\right) \;. \tag{B.25}$$

We insert the integral (B.24) into the generating function (4.87) and exchange the F and B with the A integration. The resulting F and B integrals are Gaussian, leading to

$$\int \mathrm{d}[F] \; P(F|C_F) \exp\left(\imath \mathrm{tr} FF^{\dagger}M\right) = \det{}^{-n_1/\gamma_1}\left(\mathbf{1}_{\gamma_2 p} - \imath C_F M\right) \;, \tag{B.26}$$

and likewise for $BB^{\dagger}$ with n_1 replaced by n_2. Here M is a $p \times p$ matrix in the same symmetry class as $FF^{\dagger}$. If we average the A integrand with respect to B and F using (B.26), the generating function (4.87) becomes

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \mathrm{sdet}^{-p}\left(\mathbf{1}_{2k} + \kappa\right) \int \mathrm{d}[A] \; \det{}^{-n_1/\gamma_1}\left(\mathbf{1}_{\gamma_2 p} - \imath A\mathbf{j}A^{\dagger}\right) \\ \times \det{}^{-n_2/\gamma_1}\left(\mathbf{1}_{\gamma_2 p} - \imath \hat{\Lambda}_{\text{eff}}AA^{\dagger}\right) \;, \end{aligned} \tag{B.27}$$

The integral (B.27) depends on invariants $A\mathbf{j}A^\dagger$ and $\hat{\Lambda}_{\text{eff}}AA^\dagger$ only. By means of the trace duality (2.110), these are in one-to-one corresponds with the superinvariants of $A^\dagger A\mathbf{j}$, respectively, $A^\dagger\hat{\Lambda}_{\text{eff}}A$, yielding

$$\begin{aligned}Z_p^{k/k}(\kappa) = K\text{sdet}^{-p}\left(\mathbf{1}_{2k}+\kappa\right)\int \text{d}[A]\ \text{sdet}^{-n_1/\gamma_1}\left(\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}-\imath A^\dagger A\mathbf{j}\right)\\ \times\,\text{sdet}^{-n_2/\gamma_1}\left(\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}-\imath A^\dagger\hat{\Lambda}_{\text{eff}}A\right)\ ,\end{aligned} \tag{B.28}$$

where $A^\dagger\hat{\Lambda}_{\text{eff}}A$ and $A^\dagger A$ are $(\tilde{\gamma}k|\tilde{\gamma}k)\times(\tilde{\gamma}k|\tilde{\gamma}k)$-dimensional. The main difference to models discussed in the literature so far is that Eq. (B.28) involves two different products of A and $A^\dagger$. Namely, $A^\dagger A$ which arises if invariant matrix models are considered and $A^\dagger\hat{\Lambda}_{\text{eff}}A$. The latter is due to a non-trivial correlation structure in the generating function (4.87). We therefore apply the generalized Hubbard-Stratonovich transformation to replace $A^\dagger A$ and $A^\dagger\hat{\Lambda}_{\text{eff}}A$ separately. As a consequence, we obtain an average including two supermatrices given by

$$\begin{aligned}Z_{p,n_1,n_2,\beta}^{k/k}(\kappa) = K\text{sdet}^{-p}\left(\mathbf{1}_{2k}+\kappa\right)\int \text{d}[\sigma]\ \text{d}[\varrho]\ I_{n_2}(\varrho)I_{n_1}(\sigma)\exp\left(-\text{str}\varrho-\text{str}\sigma\right)\\ \times\,\text{sdet}^{-1/\tilde{\gamma}}\left(\mathbf{1}_p\otimes\sigma-\Lambda_{\text{eff}}\otimes\varrho\mathbf{j}\right)\ ,\end{aligned} \tag{B.29}$$

where the function $I_{n_i}(\varrho)$, $i=1,2$, is the supersymmetric Ingham-Siegel integral (2.97) and ρ and σ are as explained in section 2.4.3.

B.6 Equivalence of Two-Wishart and Lorentz Matrix Model

In an exact calculation we show the equivalence of the two-Wishart matrix average (4.92) and the Lorentzian distributed generating function (4.93). We use the invariance of the generating function (4.92) under the right action of G_{n_2} on B and replace B by $\hat{B}$, where $\hat{B}$ is a square $p\times p$ matrix. This coordinate change induces a decomposition of the volume element

$$\text{d}[B]\ \sim\det{}^{\nu_2/\gamma_1}\hat{B}^T\hat{B}\ \text{d}[\hat{B}]\ , \tag{B.30}$$

where we introduce the rectangularity $\nu_i=n_i-p$. We find a similar decomposition when we replace F by $\hat{F}$, a square $p\times p$ matrix such that the generating function becomes

$$\begin{aligned}Z_p^{k/k}(\kappa) = K\int \text{d}[\hat{F},\hat{B}]\ \text{sdet}^{-1}\left[\frac{\hat{F}\hat{F}^\dagger-\hat{B}\hat{B}^\dagger}{\hat{F}\hat{F}^\dagger+\hat{B}\hat{B}^\dagger}\otimes\mathbf{1}_{k|k}-\mathbf{1}_{\gamma_2 p}\otimes\kappa\right]\\ \times\det{}^{\nu_2/\gamma_1}\hat{B}^T\hat{B}\ \det{}^{\nu_1/\gamma_1}\hat{F}^T\hat{F}\ P(\hat{F}|\mathbf{1}_{\gamma_2 p})P(\hat{B}|\hat{\Lambda}_{\text{eff}})\ .\end{aligned} \tag{B.31}$$

In the integral (B.31), we apply a singular value decomposition to $\hat{B}=U\hat{b}V^\dagger$, where $U,V\in\text{G}_p$, $\hat{b}=\mathbf{1}_{\gamma_2}\otimes b$ and $b=\text{diag}(b_1,\ldots,b_p)$ is the diagonal matrix of the distinct positive definite singular values of B. The coordinate change induces a decomposition of the volume element given by [44]

$$\mathrm{d}[\hat{B}] \ \sim \left|\Delta_p(bb^T)\right|^{\beta} \mathrm{d}[b]\ \mathrm{d}\mu(U)\mathrm{d}\mu(V)\ . \tag{B.32}$$

Since the generating function (B.31) depends on $\hat{B}\hat{B}^\dagger$ only, the V integral is trivial and is absorbed into the overall constant K. We substitute everything into the generating function (B.23) and are left with

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int \mathrm{d}[b]\ \mathrm{d}[\hat{F}]\ \det{}^{\nu_1/\gamma_1}\hat{F}\hat{F}^\dagger \left|\Delta_p(bb^T)\right|^{\beta} \det{}^{\beta\nu_2/2} bb^T \\ \times \int \mathrm{d}\mu(U) \exp\left(-\frac{\beta}{2}\mathrm{tr} U\hat{b}\hat{b}^T U^\dagger \hat{\Lambda}_{\text{eff}}^{-1}\right) \exp\left(-\frac{\beta}{2}\mathrm{tr}\hat{F}\hat{F}^\dagger\right) \\ \times\ \mathrm{sdet}^{-1}\left[\frac{\hat{F}\hat{F}^\dagger - \hat{b}\hat{b}^T}{\hat{F}\hat{F}^\dagger + \hat{b}\hat{b}^T} \otimes \mathbf{1}_{k|k} - \mathbf{1}_{\gamma_2 p}\otimes\kappa\right]\ . \end{aligned} \tag{B.33}$$

In account of $bb^T = b^T b$, we replace $\hat{b}\hat{b}^T$ in the superdeterminant by $\hat{b}^T\hat{b}$ and return to the full $\hat{B}$ space. In other words, we perform the steps leading from Eq. (B.31) to Eq. (B.33) in the backward direction. We replace in the expression arising thereby $\hat{B}$ by $\hat{B}\hat{F}^\dagger$ such that F drops out of the superdeterminant and find

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int \mathrm{d}[\hat{B}]\ \det{}^{\nu_2/\gamma_2}\hat{B}\hat{B}^\dagger\ \mathrm{sdet}^{-1}\left[\frac{\mathbf{1}_{\gamma_2 p} - \hat{B}\hat{B}^\dagger}{\mathbf{1}_{\gamma_2 p} + \hat{B}\hat{B}^\dagger}\otimes\mathbf{1}_{k|k} - \mathbf{1}_{\gamma_2 p}\otimes\kappa\right] \\ \times \int \mathrm{d}[\hat{F}]\ \det{}^{(\nu_1+\nu_2+p)/\gamma_2}\hat{F}\hat{F}^\dagger \exp\left(-\frac{\beta}{2}\mathrm{tr}\hat{F}\left[\hat{B}^\dagger\hat{\Lambda}_{eff}^{-1}\hat{B} + \mathbf{1}_{\gamma_2 p}\right]\hat{F}^\dagger\right)\ . \end{aligned} \tag{B.34}$$

By the last coordinate change, we reduce the $\hat{F}$ to a Gaussian integral, yielding

$$\begin{aligned} \int \mathrm{d}[\hat{F}]\ \det{}^{(\nu_1+\nu_2+p)/\gamma_2}\hat{F}\hat{F}^\dagger \exp\left(-\frac{\beta}{2}\mathrm{tr}\hat{F}\left[\hat{B}^\dagger\hat{\Lambda}_{eff}^{-1}\hat{B} + \mathbf{1}_{\gamma_2 p}\right]\hat{F}^\dagger\right) \\ \sim \det{}^{-(n_1+n_2)/\gamma_1}\left(\hat{B}^\dagger\hat{\Lambda}_{eff}^{-1}\hat{B} + \mathbf{1}_{\gamma_2 p}\right)\ . \end{aligned} \tag{B.35}$$

To apply the projection formula in the next section, it is necessary to provide an invariant distribution function. Thus, we perform the last coordinate change $\hat{B} \to \sqrt{\Lambda_{eff}}B$, where B is the original matrix of dimension $p\times n_2$, such that the generating function (4.87) becomes

$$\begin{aligned} Z_p^{k/k}(\kappa) = K \int \mathrm{d}[B]\ \mathrm{sdet}^{-1}\left[\frac{\Lambda_{eff}^{-1} - BB^\dagger}{\Lambda_{eff}^{-1} + BB^\dagger}\otimes\mathbf{1}_{k|k} - \mathbf{1}_{\gamma_2 p}\otimes\kappa\right] \\ \times \det{}^{-n/\gamma_1}\left(B^\dagger B + \mathbf{1}_{\gamma_2 n_2}\right)\ , \end{aligned} \tag{B.36}$$

where $n = n_1 + n_2$.

B.7 Derivation of Eq. (4.95)

We express the superdeterminant in Eq. (4.94) as a Gaussian integral over a supervector, see Eq. (2.70), which we rearrange, as we did in appendix B.5, to obtain

$$\begin{aligned}&\mathrm{sdet}^{-1}\left[\hat{\Lambda}_{\mathrm{eff}}^{-1}\otimes\frac{\mathbf{1}_{2k}-\kappa}{\mathbf{1}_{2k}+\kappa}-BB^{\dagger}\otimes\mathbf{1}_{2k}\right]\\ &\sim\int d[A]\ \exp\left(\imath\mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}^{-1}A\mathbf{j}A^{\dagger}-\imath\mathrm{tr}BB^{\dagger}AA^{\dagger}\right)\ ,\end{aligned}\tag{B.37}$$

where $\mathbf{j}$ is given by Eq. (B.25) and A as in Eq. (2.81), (2.76) and (2.82) for $\beta = 1, 2, 4$, respectively. We substitute the Gaussian integral (B.37) into the generating function (4.94) and exchange the A and the B integral and find

$$\begin{aligned}Z_p^{k/k}(\kappa)=&K\int d[A]\ \exp\left(\imath\mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}A\mathbf{j}A^{\dagger}\right)\\ &\times\int d[B]\ \exp\left(-\imath\mathrm{tr}BB^{\dagger}AA^{\dagger}\right)\mathrm{det}^{-n/\gamma_1}\left(B^{\dagger}B+\mathbf{1}_{\gamma_2 n_2}\right)\ .\end{aligned}\tag{B.38}$$

The integral above has a similar form as the matrix model (4.59). We therefore extend, as explained there, the domain of integration from the space of $p\times n_2$ matrices B to the space of rectangular $(p+k\tilde{\gamma}|k\tilde{\gamma})\times(n_1|0)$ supermatrices Σ, where

$$\Sigma=\begin{pmatrix}B\\(w_{ia}^*)\\(w_{ia})\\(-\xi_{ia}^*)\\(\xi_{ia})\end{pmatrix}\ ,\ \Sigma=\begin{pmatrix}B\\(w_{ia}^*)\\(-\xi_{ia}^*)\end{pmatrix}\ \text{and}\ \ \Sigma=\begin{pmatrix}\multicolumn{2}{c}{B}\\(w_{ia}^*)&(\tilde{w}_{ia}^*)\\(-\tilde{w}_{ia})&(w_{ia})\\(-\xi_{ia}^*)&(-\tilde{\xi}_{ia}^*)\\(-\tilde{\xi}_{ia})&(\xi_{ia})\end{pmatrix}\tag{B.39}$$

for $\beta = 1, 2, 4$, respectively. This extension leads to the following form of the generating function

$$Z_p^{k/k}(\kappa)=K\int d[A]\ \frac{\exp\left(\imath\mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}^{-1}A\mathbf{j}A^{\dagger}\right)}{\mathrm{sdet}^p\left(\mathbf{1}_{2k}+\kappa\right)}\int d[\Sigma]\ \frac{\exp\left(-\imath\mathrm{str}\Sigma\Sigma^{\dagger}\left[\begin{array}{cc}AA^{\dagger}&0\\0&0\end{array}\right]\right)}{\mathrm{det}^{n/\gamma_1}\left(\Sigma^{\dagger}\Sigma+\mathbf{1}_{n_2\gamma_2}\right)}\tag{B.40}$$

$$=K\int d[A]\ \frac{\exp\left(\imath\mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}^{-1}A\mathbf{j}A^{\dagger}\right)}{\mathrm{sdet}^p\left(\mathbf{1}_{2k}+\kappa\right)}\phi\left(\left[\begin{array}{cc}AA^{\dagger}&0\\0&0\end{array}\right]\right)\ ,\tag{B.41}$$

where ϕ is the Fourier transform with respect to the Lorentzian distribution. Because the latter is invariant under the right action of the proper symmetry group, the former is invariant under the adjoint action G_n of the same group. Thus, by the arguments given below Eq. (4.67), we can apply the trace duality (2.110) and write Eq. (B.41) as

$$Z_p^{k/k}(\kappa)=K\int d[A]\ \frac{\exp\left(\imath\mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}^{-1}A\mathbf{j}A^{\dagger}\right)}{\mathrm{sdet}^p\left(\mathbf{1}_{2k}+\kappa\right)}\phi\left(\left[\begin{array}{cc}0&0\\0&A^{\dagger}A\end{array}\right]\right)\tag{B.42}$$

$$\begin{aligned}&= K\int d[A]\ \frac{\exp\left(\imath \mathrm{tr}\hat{\Lambda}_{\mathrm{eff}}^{-1}A\mathbf{j}A^\dagger\right)}{\mathrm{sdet}^p\left(\mathbf{1}_{2k}+\kappa\right)}\\ &\times\int d[\Sigma]\ \frac{\exp\left(-\imath \mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc}0&0\\0&A^\dagger A\end{array}\right]\right)}{\det^{n/\gamma_1}\left(\Sigma^\dagger\Sigma+\mathbf{1}_{n_2\gamma_2}\right)}\ ,\end{aligned}\tag{B.43}$$

where

$$\mathrm{str}\Sigma\Sigma^\dagger\left[\begin{array}{cc}0&0\\0&A^\dagger A\end{array}\right]=\mathrm{str}\chi\chi^\dagger A^\dagger A\ .\tag{B.44}$$

Hence, only the determinant in the denominator of Eq. (B.43) depends on B. Moreover, contrary to Eq. (B.41), the generating function (B.43) depends on the $(k\tilde{\gamma}|k\tilde{\gamma})\times(k\tilde{\gamma}|k\tilde{\gamma})$ matrix $A^\dagger A$ only.

The rectangular supermatrix χ as introduced in Eq. (B.43), is the $(k\tilde{\gamma}|k\tilde{\gamma})\times(n_1|0)$-dimensional part of Σ "below" B in Eq. (B.39) and reads

$$\chi=\begin{pmatrix}(w_{ia}^*)\\(w_{ia})\\(-\xi_{ia}^*)\\(\xi_{ia})\end{pmatrix}\ ,\ \chi=\begin{pmatrix}(w_{ia}^*)\\(-\xi_{ia}^*)\end{pmatrix}\ \text{and}\ \ \chi=\begin{pmatrix}(w_{ia}^*)&(\tilde{w}_{ia}^*)\\(-\tilde{w}_{ia})&(w_{ia})\\(-\xi_{ia}^*)&(-\tilde{\xi}_{ia}^*)\\(-\tilde{\xi}_{ia})&(\xi_{ia})\end{pmatrix}\tag{B.45}$$

for $\beta=1,2,4$, respectively. We decompose the measure

$$d[\Sigma]\ = d[\chi]\ d[B]\ \ ,\tag{B.46}$$

and exchange the A and the χ integral. The resulting A integral is a Gaussian integral. The B integral obtained thereby was computed in Ref. [140] and is given by

$$\int d[B]\ \det^{-n/\gamma_1}\left(B^\dagger B+\chi^\dagger\chi+\mathbf{1}_{n_2\gamma_2}\right)\sim \mathrm{sdet}^{-(n-p)/\gamma_1}\left(\chi\chi^\dagger+\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}\right)\ ,\tag{B.47}$$

where we used that $\Sigma^\dagger\Sigma=B^\dagger B+\chi^\dagger\chi$. If we insert both into the generating function (B.43), we arrive at a representation of the generating function (4.87) in terms of an average over a $(k\tilde{\gamma}|k\tilde{\gamma})\times(k\tilde{\gamma}|k\tilde{\gamma})$-dimensional dyadic supermatrix $\chi\chi^\dagger$,

$$\begin{aligned}Z_p^{k/k}(\kappa)&=K\int d[\chi]\ \mathrm{sdet}^{-(n-p)/\gamma_1}\left(\chi\chi^\dagger+\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}\right)\\ &\times\mathrm{sdet}^{-1/\gamma_1}\left(\Lambda_{\mathrm{eff}}^{-1}\otimes(\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}-\hat{\kappa})-\mathbf{1}_p\otimes(\mathbf{1}_{\tilde{\gamma}k|\tilde{\gamma}k}+\hat{\kappa})\chi\chi^\dagger\right)\ ,\end{aligned}\tag{B.48}$$

where $\hat{\kappa}=\mathrm{diag}(\mathbf{1}_{\tilde{\gamma}}\otimes\kappa_1,\mathbf{1}_{\tilde{\gamma}}\otimes\kappa_2)$.

B.8 Performing the $\varepsilon \to 0$ Limit

This section is devoted to a detailed discussion of the $\varepsilon \to 0$ limit in section 4.3.4. To begin, we separate the integral (4.121) into its three parts such that

$$\begin{aligned}
\frac{\partial Z_p^{1/1}(x,x_0)}{\partial x} &= \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 \frac{g_1(r_1,r_1)}{\prod_{k=1}^p \sqrt{\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right)\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_2\right)}} \\
&+ \sum_{l=1}^p \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 \frac{g_{2,l}(r_1,r_1)}{\prod\limits_{k=1}^p \sqrt{\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right)\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_2\right)}} \frac{1}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)} \\
&+ \sum_{\substack{i,l=1 \\ i\neq l}}^p \int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 \frac{g_{3,l,i}(r_1,r_2)}{\prod\limits_{k',k=1}^p \sqrt{\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right)\left(\frac{1-x_0}{1+x_0}\Lambda_{k'}^{-1} - r_2\right)}} \\
&\times \frac{1}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)\left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)} \ ,
\end{aligned} \tag{B.49}$$

where g_1, $g_{2,l}$ and $g_{3,l,i}$ can be read off from Eq. (4.121). Since we are dealing with square roots of complex numbers, a branch cut has to be fixed. We choose it to lie on the negative real line. Taking the $\varepsilon \to 0$ limit is accomplished as follows. We separately consider the terms proportional to g_1, $g_{2,l}$ and $g_{3,l,i}$ in Eq. (B.49). First we study the contributions proportional to g_1. We split the expressions under the square roots into a radial and an angular part. In account of the $\varepsilon \ll 1$, we expand the latter in ε, leading to

$$\begin{aligned}
\frac{1}{\prod_{k=1}^p \sqrt{\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right)}} &= \frac{1}{\prod_{k=1}^p \sqrt{\left|\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right|}} \\
\times &\begin{cases} e^{-\imath\pi \mathrm{sign}(\mathrm{Im}(x_0)) + \mathcal{O}(\varepsilon)} & , \mathrm{Re}\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1} - r_1\right) < 0 \\ e^{\mathcal{O}(\varepsilon)} & , \mathrm{else} \end{cases}
\end{aligned} \tag{B.50}$$

and likewise for r_2. The real part of each singularity is negative if

$$\frac{\Lambda_k^{-1} - r}{\Lambda_k^{-1} + r} < x \quad \Leftrightarrow r > \Lambda_k^{-1} \frac{1-x}{1+x} \ . \tag{B.51}$$

Thus, to see whether a particular singularity has a positive or negative real part, we assume that the eigenvalues are ordered, *i.e.* $0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_p < \infty$ and divide the domain of integration $[0,\infty)$ into disjoint subsets

$$[0,\infty) = \bigcup_{j=0}^p \mathrm{V}_i \ , \tag{B.52}$$

where

$$\mathsf{V}_0 = \left[0, \Lambda_p^{-1}\frac{1-x}{1+x}\right) \ , \quad \mathsf{V}_p = \left(\Lambda_1^{-1}\frac{1-x}{1+x}, \infty\right) \tag{B.53}$$

and

$$\mathsf{V}_i = \left(\Lambda_{p-i+1}^{-1}\frac{1-x}{1+x}, \Lambda_{p-i}^{-1}\frac{1-x}{1+x}\right) \quad \text{for } i = 1, \ldots, p-1 \ . \tag{B.54}$$

Inserting this into the integrals in Eq. (B.49) corresponds to

$$\int_0^\infty \mathrm{d}r_1 \mathrm{d}r_2 \ldots \mapsto \sum_{\substack{l_1=0\\l_2=0}}^{p} \int_{\mathsf{V}_{l_1}\times\mathsf{V}_{l_2}} \mathrm{d}r_1 \mathrm{d}r_2 \ldots \tag{B.55}$$

Thus, if $r_1 \in \mathsf{V}_{l_1}$ and $r_2 \in \mathsf{V}_{l_2}$, we find using Eq. (B.55) that

$$\begin{aligned} &\operatorname{Im}\lim_{\varepsilon\to 0}\prod_{k=1}^{p}\frac{1}{\sqrt{\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1}-r_1\right)\left(\frac{1-x_0}{1+x_0}\Lambda_k^{-1}-r_2\right)}} \\ &= \begin{cases} \prod_{k=1}^{p}\frac{(-1)^{(l_1+l_2-1)/2}}{\sqrt{\left|\frac{1-x}{1+x}\Lambda_k^{-1}-r_1\right|\left|\frac{1-x}{1+x}\Lambda_k^{-1}-r_2\right|}} & , l_1+l_2 \in 2\mathbb{N}+1 \\ 0 & ,\text{else} \end{cases} , \end{aligned} \tag{B.56}$$

such that because of the imaginary part only those combinations of l_1 and l_2 contribute whose sum is odd.

The remaining terms proportional to $g_{2,l}$ and $g_{3,l,j}$ in the generating function (B.49) have non-integrable singularities when taking the $\varepsilon \to 0$ limit. This means, they have singularities to a power larger than one. The $\varepsilon \to 0$ limit then yields the principal value. Analogously to Eq. (B.55), we split the integrals over $g_{2,l}$ and $g_{3,l,j}$ into sums of integrals over the domain $\mathsf{V}_{l_1} \times \mathsf{V}_{l_2}$. For terms proportional to $g_{2,l}$, all integrals obtained thereby are handled with the analysis leading to Eq. (B.56) unless $l_1 \neq p-l, p-l+1$. To do the integrals over domains with $l_1 = p-l, p-l+1$, we decrease the order of the singularity using partial integration and a regularization term. To begin with, we consider an arbitrary function $f(r_1, r_2)$ integrable on the domain

$$\widetilde{\mathsf{V}}_l = \left(\Lambda_{l-1}^{-1}\frac{1-x}{1+x}, \Lambda_{l+1}^{-1}\frac{1-x}{1+x}\right) \ . \tag{B.57}$$

We are aiming to compute

$$J_1 = \int_0^\infty \mathrm{d}r_2 \int_{\widetilde{V}_l} \mathrm{d}r_1 \frac{f(r_1, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{3/2}} \ , \tag{B.58}$$

which exists, because of the imaginary increment of x_0. To take the $\varepsilon \to 0$ limit, we add to the integrand a zero

$$f(r_1, r_2) = f(r_1, r_2) - f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2) + f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2) \tag{B.59}$$

and find using partial integration

$$\begin{aligned} J_1 = &\int_0^\infty \mathrm{d}r_2 \int_{\Lambda_{l+1}^{-1}(1-x)/(1+x)}^{\Lambda_{l-1}^{-1}(1-x)/(1+x)} \mathrm{d}r_1 \frac{f(r_1, r_2) - f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{3/2}} \\ &+ 2\int_0^\infty \mathrm{d}r_2 \left. \frac{f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{1/2}} \right|_{\Lambda_{l+1}^{-1}\frac{1-x}{1+x}}^{\Lambda_{l-1}^{-1}\frac{1-x}{1+x}} . \end{aligned} \tag{B.60}$$

The resulting integrals in Eq. (B.60) have only integrable singularities, because if we expand $f(r_1, r_2)$ around the $3/2$ singularity in r_1, we find

$$f(r_1, r_2) = f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2) + \frac{\partial f}{\partial r_1}\left(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2\right)\left(\Lambda_l^{-1}\frac{1-x}{1+x} - r_1\right) + \mathcal{O}(r_1^2) . \tag{B.61}$$

If we substitute the expansion (B.61) into the integrand in the first row of Eq. (B.60) the $\mathcal{O}(r_1^0)$ terms cancels each other, yielding

$$\frac{f(r_1, r_2) - f(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{3/2}} \approx -\frac{\frac{\partial f}{\partial r_1}(\Lambda_l^{-1}\frac{1-x}{1+x}, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{1/2}} . \tag{B.62}$$

Thus, for r_1 in the vicinity of the singularity, the expansion turns out to be integrable. Analogous, we can treat the third term in Eq. (B.49). For terms proportional to $g_{3,l,i}$ all singularities are integrable as long as we integrate the domains $\mathsf{V}_{l_1} \times \mathsf{V}_{l_2}$ with $l_1 \neq p-l, p-l+1$ and $l_2 \neq p-i, p-i+1$. On the remaining domains where $l_1 = p-l, p-l+1$ and $l_2 \neq p-i, p-i+1$, $l_1 \neq p-l, p-l+1$ and $l_2 = p-i, p-i+1$ and $l_1 = p-l, p-l+1$ and $l_2 = p-i, p-i+1$ the integrand has non-integrable singularities which we study using a regularization. We consider again $f(r_1, r_2)$ to be an arbitrary integrable function and make use of the following scheme

$$\int_{\tilde{\mathsf{V}}_l \times \tilde{\mathsf{V}}_i} \mathrm{d}r_1 \mathrm{d}r_2 \frac{f(r_1, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{3/2}\left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)^{3/2}}$$

$$
\begin{aligned}
&= \int\limits_{\widetilde{\mathrm{V}}_l \times \widetilde{\mathrm{V}}_i} \mathrm{d}r_1 \mathrm{d}r_2 \frac{f(r_1,r_2) + f(\frac{1-x}{1+x}\Lambda_l^{-1}, \frac{1-x}{1+x}\Lambda_i^{-1}) - f(\frac{1-x}{1+x}\Lambda_l^{-1}, r_2) - f(r_1, \frac{1-x}{1+x}\Lambda_i^{-1})}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{3/2} \left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)^{3/2}} \\
&-4 \left. \frac{f(\frac{1-x}{1+x}\Lambda_l^{-1}, \frac{1-x}{1+x}\Lambda_i^{-1}) - f(\frac{1-x}{1+x}\Lambda_l^{-1}, r_2) - f(r_1, \frac{1-x}{1+x}\Lambda_i^{-1})}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{1/2} \left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)^{1/2}} \right|_{(r_1,r_2)\in\partial(\widetilde{\mathrm{V}}_l \times \widetilde{\mathrm{V}}_i)} \\
&-2 \int\limits_{\widetilde{\mathrm{V}}_j} \mathrm{d}r_1 \left. \frac{\partial_{r_1} f(r_1, \frac{1-x}{1+x}\Lambda_i^{-1})}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{1/2} \left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)^{1/2}} \right|_{r_2\in\partial\widetilde{\mathrm{V}}_i} \\
&-2 \int\limits_{\widetilde{\mathrm{V}}_i} \mathrm{d}r_2 \left. \frac{\partial_{r_2} f(\frac{1-x}{1+x}\Lambda_l^{-1}, r_2)}{\left(\frac{1-x_0}{1+x_0}\Lambda_l^{-1} - r_1\right)^{1/2} \left(\frac{1-x_0}{1+x_0}\Lambda_i^{-1} - r_2\right)^{1/2}} \right|_{r_1\in\partial\widetilde{\mathrm{V}}_l} .
\end{aligned}
\tag{B.63}
$$

In Eq. (B.63), we make use of the shorthand notation $\partial_{r_1} = \partial/\partial r_1$ and introduce ∂M as the boundary of M, *e.g.* $\partial\widetilde{\mathrm{V}}_j = \left\{\Lambda_{j+1}^{-1}(1-x)/(1+x)\ ,\Lambda_{j-1}^{-1}(1-x)/(1+x)\right\}$. Substituting Eq. (B.63) into the threefold sum in Eq. (B.49), the resulting integrals have only integrable singularities. Thus, we can apply the analysis leading to Eq. (B.56) and finally arrive at

$$
\begin{aligned}
S(x) = &\sum_{\substack{0\le l_1,l_2\le p \\ l_1+l_2\in 2\mathbb{N}+1}} \int\limits_{\mathrm{V}_{l_1}\times\mathrm{V}_{l_2}} \mathrm{d}r_1 \mathrm{d}r_2 \frac{(-1)^{(l_1+l_2-1)/2} g_1(r_1,r_2)}{\prod_{k=1}^{p}\sqrt{\left|\frac{1-x}{1+x}\Lambda_k^{-1} - r_1\right| \left|\frac{1-x}{1+x}\Lambda_k^{-1} - r_2\right|}} \\
&\left(1 + \sum_{l=1}^{p} \frac{f_{2,l}(r_1,r_1)}{\left(\frac{1-x}{1+x}\Lambda_l^{-1} - r_1\right)} + \sum_{\substack{i,l=1 \\ i\neq l}}^{p} \frac{f_{3,l,i}(r_1,r_1)}{\left(\frac{1-x}{1+x}\Lambda_l^{-1} - r_1\right)\left(\frac{1-x}{1+x}\Lambda_i^{-1} - r_2\right)}\right) ,
\end{aligned}
\tag{B.64}
$$

where $f_{2,l}(r_1,r_1)$ and $f_{3,l,i}(r_1,r_1)$ are readily determined from Eqs. (4.121), (B.60) and (B.63) .

APPENDIX C

Supplemental Material Chapter 5

C.1 Known Results for the Complex Ensemble

This section is devoted to a detailed computation of the gap probability corresponding to the largest eigenvalue distribution found in Ref. [143]. We start our discussion at Eq. (2.37). For $\beta = 2$ the Harish-Chandra-Itzykson-Zuber integral is given by [95,96]

$$\Phi_2(X, \Lambda^{-1}) = \int \mathrm{d}\mu(U) \exp\left(-\mathrm{tr} U X U^\dagger \Lambda^{-1}\right) \sim \frac{\det\left[\exp(-x_i/\Lambda_j)\right]}{\Delta_p(X)\Delta_p(\Lambda^{-1})} , \tag{C.1}$$

where the integral is over the unitary group $\mathrm{U}(n)$, $\Delta_p(X) = \prod_{i<j}(x_i - x_j)$ and $i, j = 1, \ldots, p$. Substituting this into Eq. (2.37) and using $\Delta_p(\Lambda^{-1}) \sim \det^{p-n}\Lambda\Delta_p(\Lambda)$ and the invariance of the integrand under permutation of the integration variables, we write the gap probability as

$$\begin{aligned} E_p^{(\beta)}([0,t];p) = \frac{K}{\det^{p-n}\Lambda\Delta_p(\Lambda)} \int_0^t \mathrm{d}x_1 \cdots \int_0^t \mathrm{d}x_p \Delta_p(X) \\ \times \det^{n-p} X \exp\left(-\mathrm{tr} X \Lambda^{-1}\right) . \end{aligned} \tag{C.2}$$

The remaining eigenvalue integral is of standard type and can be solved in various ways. We write the determinant as sum over permutations and exchange this sum with the integrals. The expression arising thereby is determinantal with a kernel in terms of a single integral,

$$E_p^{(\beta)}([0,t];p) = \frac{K}{\det^{p-n}\Lambda\Delta_p(\Lambda)} \det\left[\int_0^t \mathrm{d}x x^{n-p+i-1} \exp\left(-x/\Lambda_j\right)\right] , \tag{C.3}$$

where $1 \leq i, j \leq p$. The integral in the matrix kernel of the determinant is rather simple and given by

$$\int_0^t \mathrm{d}x x^m e^{-x/\Lambda_j} = (-1)^m m! \Lambda_j^{m+1} \left(1 - \sum_{l=0}^{m} \frac{\exp\left(-t/\Lambda_j\right)}{l!} \left(\frac{t}{\Lambda_j}\right)^l\right) . \tag{C.4}$$

We substitute it into Eq. (C.3), employ $E_p^{(2)}([0,t];p) \to 1$ for $t \to \infty$ and arrive at

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{(-1)^{p(p-1)/2}}{\Delta_p(\Lambda)} \\ \times \det &\left[\Lambda_j^{i-1} \left(1 - \sum_{m=0}^{n-p+i-1} \frac{\exp(-t/\Lambda_j)}{m!} \left(\frac{t}{\Lambda_j}\right)^m \right) \right] , \end{aligned} \tag{C.5}$$

where $i, j = 1, \ldots, p$.

C.2 First Derivation in the Complex Case

In this this section we give the details of the derivation of Eq. (5.17). We employ the results of Ref. [111] and cast the whole integrand (5.15) into a product of two determinants,

$$\frac{\Delta_n(\imath Y + \mathbf{1}_n)}{\prod_{k,i}^{p,n}\left(t\Lambda_k^{-1} + \imath y_i + 1\right)} \sim \frac{1}{\Delta_p(t\Lambda^{-1})} \det\left[\frac{1}{\left(t\Lambda_k^{-1} + \imath y_i + 1\right)} \middle| (\imath y_i + 1)^{l-1} \right] , \tag{C.6}$$

$$\frac{\Delta_n(\imath Y + \mathbf{1}_n) \prod_{i=1}^n \exp(\imath y_i + 1)}{\prod_{i=1}^n (\imath y_i + 1)^n} \sim \det\left[\frac{\exp(\imath y_i + 1)}{(\imath y_i + 1)^{n-i'+1}} \right] , \tag{C.7}$$

where $k = 1, \ldots, p$, $1 \le i, i' \le n$ and $l = 1, \ldots, n-p$ such that the matrices in the determinants are of dimension $n \times n$. We substitute the determinants into Eq. (5.15), apply Andréjif's integral theorem (c.f. appendix of Ref. [111]) and find

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{Kt^{pn}}{\Delta_p(t\Lambda^{-1})} \\ \det &\left[\int\limits_{-\infty}^{\infty} \mathrm{d}z \frac{\exp(\imath z + 1)\,(\imath z + 1)^{i'-n-1}}{\left(t\Lambda_k^{-1} + \imath z + 1\right)} \middle| \int\limits_{-\infty}^{\infty} \mathrm{d}z \frac{\exp(\imath z + 1)}{(\imath z + 1)^{n-l-i'+2}} \right] . \end{aligned} \tag{C.8}$$

In both determinant kernels, the poles of the integrand lie in the upper complex half-plane. Because of the exponential, we write the kernels as contour integrals, with a contour closed in upper half-plane. The contour has to enclose all poles of the integrand. After a shift of the resulting integral contour $z \to z + \imath$, the gap probability (C.8) becomes

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{Kt^{p(2n-p+1)/2}}{\Delta_p(\Lambda^{-1})} \\ \det &\left[\oint\limits_{\mathcal{C}_1} \mathrm{d}z \frac{(\imath z)^{n-i'+1}\exp(\imath z)}{\left(t\Lambda_k^{-1} + \imath z\right)} \middle| \oint\limits_{\mathcal{C}_2} \mathrm{d}z \frac{\exp(\imath z)}{(\imath z)^{n-l-i'+2}} \right] , \end{aligned} \tag{C.9}$$

where $\mathcal{C}_i$ enclose all poles of the integrand. The remaining integrals are covered by the residue theorem, leading to

$$\oint\limits_{\mathcal{C}_2} \mathrm{d}z \frac{\exp(\imath z)}{(\imath z)^{n-l-i'+2}} = \begin{cases} \dfrac{2\pi}{(n-i'-l+1)!} & , n \ge i' + l - 1 \\ 0 & , \text{else} \end{cases} \tag{C.10}$$

and

$$\begin{aligned}&\oint_{\mathcal{C}_1} \mathrm{d}z \frac{\exp(\imath z)(\imath z)^{i'-n-1}}{\left(t\Lambda_k^{-1}+\imath z\right)} \\ =&2\pi\left(\left(\frac{-\Lambda_k}{t}\right)^{n-i'+1}\exp\left(-\frac{t}{\Lambda_k}\right)-\sum_{m=0}^{n-i'}\frac{(-\Lambda_k/t)^{m+1}}{(n-i'-m)!}\right)\ .\end{aligned} \tag{C.11}$$

The latter expression is of a similar form as the matrix kernel in Eq. (5.14). However, if we substitute Eq. (C.10) and Eq. (C.11) into the determinant in Eq. (C.9), we obtain a determinant with a $n \times n$ dimensional kernel,

$$\begin{aligned}&E_p^{(2)}([0,t];p) = \frac{Kt^{p(2n-p+1)/2}}{\Delta_p(\Lambda^{-1})} \\ &\det\left[\begin{array}{c|c} \left(\frac{-\Lambda_k}{t}\right)^{n-i'+1} e^{-t/\Lambda_k} - \sum\limits_{m=0}^{n-i'} \frac{(-\Lambda_k/t)^{m+1}}{(n-i'-m)!} & \frac{1}{(n-i'-l+1)!} \\ \hline \left(\frac{-\Lambda_k}{t}\right)^{n-p-j+1} e^{-t/\Lambda_k} - \sum\limits_{m=0}^{n-p-j} \frac{(-\Lambda_k/t)^{m+1}}{(n-p-j-m)!} & \frac{\Theta(n-p-j-l+1)}{(n-p-j-l+1)!} \end{array}\right],\end{aligned} \tag{C.12}$$

where $i', k = 1, \ldots, p$ and $j, l = 1, \ldots, n-p$. For later use, we split the $n \times n$ matrix into four blocks. The blocks are $p \times p$, $p \times (n-p)$, $(n-p) \times p$ and $(n-p) \times (n-p)$ dimensional. To simplify the matrix in the determinant, we utilize the invariance of the latter under building linear combinations of column vectors. We add to each of the first p columns in the determinant, linear combination of all columns in the second block, consisting of the last $n-p$ columns. For instance, if we add the lth column of the second block to the k column of the first block, we choose the prefactor to be $(-\Lambda_k/t)^{l+1}$. This leads to

$$\begin{aligned}&E_p^{(2)}([0,t];p) = \frac{Kt^{pn-p(p-1)/2}}{\Delta_p(\Lambda^{-1})} \\ &\det\left[\begin{array}{c|c} \left(\frac{-\Lambda_k}{t}\right)^{n-i'+1} e^{-t/\Lambda_k} - \sum\limits_{m=n-p}^{n-i'} \frac{(-\Lambda_k/t)^{m+1}}{(n-i'-m)!} & \frac{1}{(n-i'-l+1)!} \\ \hline \left(\frac{-\Lambda_k}{t}\right)^{n-p-j+1} e^{-t/\Lambda_k} & \frac{\Theta(n-p-j-l+1)}{(n-p-j-l+1)!} \end{array}\right].\end{aligned} \tag{C.13}$$

For the complex case, we have seen that the gap probability (5.5) is given by a determinant of size $p \times p$, see Eq. (C.5). To reduce the $n \times n$ dimensional kernel to a $p \times p$-dimensional one, we apply the Schur complement. It states that if D is an invertible matrix, then

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det\left(A - BD^{-1}C\right)\ . \tag{C.14}$$

The lower right $(n-p)\times(n-p)$ dimensional block of the matrix (C.13), which we denote by D, has determinant ± 1. It is invertible. Applying the Schur complement (C.14) to Eq. (C.13), leads to a determinant of a $p\times p$ dimensional matrix. Because of its structure the inverse matrix of D is

$$\left[D^{-1}\right]_{lf} = \frac{(-1)^{l+f-n+p-1}}{(l+f-n+p-1)!}\Theta(l+f-n+p-1). \tag{C.15}$$

Hence, it is a triangular matrix, but it has non-zero entries only on and below the anti-diagonal. If we apply Eq. (C.14) to Eq. (C.13) with the aid of Eq. (C.15), we arrive at

$$E_p^{(2)}([0,t];p) = \frac{Kt^{pn-p(p+1)/2}}{\Delta_p(\Lambda)}\det\left[K_{i'k}^{\beta=2}(\Lambda)\right]\ , \tag{C.16}$$

where

$$\begin{aligned} K_{i'k}^{\beta=2}(\Lambda) = & \left(\frac{-\Lambda_k}{t}\right)^{n-i'}\exp\left(-\frac{t}{\Lambda_k}\right) + \sum_{m=n-p}^{n-i'}\frac{(-\Lambda_k/t)^m}{(n-i'-m)!} \\ & -\sum_{f=1}^{n-p}\left(\frac{-\Lambda_k}{t}\right)^{n-p-f}\exp\left(-\frac{t}{\Lambda_k}\right)\sum_{l=0}^{f-1}\frac{(-1)^l}{(p+f-i'-l)!}\ . \end{aligned} \tag{C.17}$$

The normalization constant K is derived by employing $E_p^{(2)}([0,t];p)\to 1$ for $t\to\infty$. In this limit, only the second sum in the first line of Eq. (C.17) contributes, because all other terms are exponentially suppressed. Thus,

$$K = \frac{(-1)^{np-p(p+1)/2}}{\det^{n-p}\Lambda}\ , \tag{C.18}$$

is the result if we use the base independence of the determinant.

C.3 Cauchy Transform of the Orthogonal Polynomial

We present a detailed derivation of the Cauchy transform of a orthogonal polynomial used in section 5.2.3. A classical result in the theory of orthogonal polynomials is that they can be expressed in terms of eigenvalue integrals [40,175]. In the present case this yields

$$R_i(z) \sim \int \mathrm{d}[Y]\ \Delta_i^2\left(\imath Y+\mathbf{1}_i\right)\prod_{j=1}^{i}\frac{(y_j-z)\exp\left(\imath y_j+1\right)}{(\imath y_j+1)^n}\ . \tag{C.19}$$

We substitute the integral (C.19) into the kernel on the right hand side of Eq. (5.21) and find that the Cauchy transform combined with the eigenvalue integral yields

$$\int_{-\infty}^{\infty}\mathrm{d}z\frac{R_{i-1}(z)\,w(z)}{\left(t\Lambda_k^{-1}+\imath z+1\right)} \sim \int\frac{\mathrm{d}[Y]\ \Delta_i^2\left(\imath Y+\mathbf{1}_i\right)\exp\left(\mathrm{tr}\left(\imath Y+\mathbf{1}_i\right)\right)}{\det^n(\imath Y+\mathbf{1}_i)\det(\imath Y+\mathbf{1}_i+t\Lambda_k^{-1}\mathbf{1}_i)}\ . \tag{C.20}$$

This can be seen by applying partial fraction decomposition to the determinant in the denominator including the empirical eigenvalue on the right hand side of Eq. (C.20) and using its invariance under permutation of the individual eigenvalue integrals. Employing the analysis of section 5.2.1 in backward direction to Eq. (C.20) yields

$$\begin{aligned}&\int\limits_{-\infty}^{\infty} \mathrm{d}z \frac{R_{i-1}(z)\, w(z)}{\left(t\Lambda_k^{-1} + \imath z + 1\right)} \\ &\sim \int \mathrm{d}[v]\ \exp\left(-\frac{t}{\Lambda_k} v v^\dagger\right) \Theta(1 - v v^\dagger)(1 - v v^\dagger)^{n-i}\end{aligned} \tag{C.21}$$

$$\sim \int\limits_0^1 \mathrm{d}x \exp\left(-\frac{t}{\Lambda_k} x\right) x^{i-1}(1-x)^{n-i} = \varphi_{i-1}^{n-i}\left(t\Lambda_k^{-1}\right)\ , \tag{C.22}$$

where v is a complex vector of dimension i. The factor $(1 - vv^\dagger)$ to a power of $n - i$ is due to the difference between the number of eigenvalues in Eq. (C.20) and the power $-n$ of $(\imath x + 1)$ in the weight function (5.16), see appendix D. If they differ, Eq. (D.4) leads to an additional determinant to a power of $n-i$ of the matrix $\mathbf{1}_i - v^\dagger v$. Since v is a i-dimensional complex vector, the determinant of $\mathbf{1}_i - v^\dagger v$ is equivalent to the scalar $(1 - vv^\dagger)$. We introduce the function

$$\varphi_s^m(z) = \sum_{l=0}^{m} \frac{m!(-1)^l}{l!(m-l)!} \left(\frac{(s+l)!}{z^{s+l+1}} - \sum_{j=0}^{s+l} \frac{(s+l)!}{j!} \frac{\exp(-z)}{z^{s+1-j}}\right) \tag{C.23}$$

for comprehensibility and clarity. It will appear several times in the main text.

C.4 Derivation of the Determinantal Expression (5.35)

This section is devoted to the derivation of the determinantal expression (5.35). First, we calculate the overall normalization constant K in Eq. (5.15). From

$$\begin{aligned}\lim_{t\to\infty} E_p^{(2)}([0,t];p) &= K\det{}^n\Lambda \int \mathrm{d}[Y]\ \Delta_n^2(Y) \prod_{i=1}^{n} \frac{\exp(\imath y_i + 1)}{(\imath y_i + 1)^n} \\ &= K(2\pi)^n n! \det{}^n \Lambda = 1\ ,\end{aligned} \tag{C.24}$$

where we make use of Eq. (D.7). We conclude that the normalization constant in Eq. (5.15) is given by

$$K = \frac{1}{(2\pi)^n n! \det^n \Lambda}\ . \tag{C.25}$$

For technical reasons, we assume that $p = 2l$ is even. For odd $p = 2l - 1$, we derive the gap probability from the even case as a special limit. We split the empirical eigenvalue matrix into two sub-matrices $\Lambda = \mathrm{diag}(\lambda, \mu)$, where $\lambda_k = \Lambda_k$

and $\mu_k = \Lambda_{k+l}$ for all $k = 1, \ldots, l$. This induces a decomposition of the product of determinants in the denominator of Eq. (5.15) into a product over λ and μ. We use the results of Ref. [111] and combine the determinants in the denominator with the Vandermonde determinants,

$$\begin{aligned}\frac{\Delta_n(Y)}{\prod_{k,i}^{l,n}(t/\lambda_k + \imath y_i + 1)} &= \frac{\imath^{nl+l+l(l-1)/2}(-1)^{n(n-1)/2}}{t^{l(l-1)/2}\text{det}^{-l+1}\lambda\Delta_l(\lambda)} \\ &\quad \times \det\left[\frac{1}{(t/\lambda_k + \imath y_i + 1)}\middle| y_i^{a-1}\right] ,\end{aligned} \tag{C.26}$$

where $1 \leq i \leq n$, $1 \leq a \leq n-l$ and $1 \leq k \leq l$ and for μ by analogy. The eigenvalue integral (5.15) reduces to a n-fold integral of Andréjif's type. Employing Andréjif's integral theorem leads to a determinantal structure with a $(n+l)\times(n+l)$ dimensional kernel,

$$E_p^{(2)}([0,t];p) = \frac{(-1)^{nl+l(l+1)/2}t^{np-l(l-1)}}{(2\pi)^n\text{det}^{n+1-l}\Lambda\Delta_l(\lambda)\Delta_l(\mu)}\det\begin{bmatrix} G(\lambda_{k_1},\mu_{k_2}) & F_a(\mu_{k_2}) \\ F_b(\lambda_{k_1}) & D_{ba}\end{bmatrix} , \tag{C.27}$$

where $1 \leq k_1, k_2 \leq l$, $1 \leq a, b \leq n-l$ and

$$G(\lambda_{k_1},\mu_{k_2}) = \int_{-\infty}^{\infty} \mathrm{d}z \frac{\exp(\imath z + 1)}{(\imath z+1)^n(t/\lambda_{k_1} + \imath z + 1)(t/\mu_{k_2} + \imath z + 1)} , \tag{C.28}$$

$$F_a(\lambda_{k_1}) = \int_{-\infty}^{\infty} \mathrm{d}z \frac{z^{a-1}\exp(\imath z + 1)}{(\imath z+1)^n(t/\lambda_{k_1} + \imath z + 1)} , \tag{C.29}$$

$$[D]_{ab} = \int_{-\infty}^{\infty} \mathrm{d}z \frac{z^{b+a-2}\exp(\imath z + 1)}{(\imath z+1)^n} . \tag{C.30}$$

The matrix D is invertible. To demonstrate this, we consider Eq. (C.27) with $l = 0$ such that only the determinant of D survives. The remaining expression is by construction proportional to Eq. (D.7), yielding

$$\text{det}D = \frac{(2\pi)^{n-l}}{(n-l)!}\prod_{j=0}^{n-l-1}\frac{(j+1)!}{(n-j-1)!} . \tag{C.31}$$

Next, we apply the Schur complement to the determinant in Eq. (C.27). The resulting expression is determinantal, with a $l\times l$ dimensional matrix kernel,

$$\begin{aligned}E_p^{(2)}([0,t];p) &= \frac{(-1)^{nl+l(l+1)/2}t^{np-l(l-1)}}{(2\pi)^n\text{det}^{n+1-l}\Lambda\Delta_l(\lambda)\Delta_l(\mu)}\text{det}D \\ &\quad \times \det\left[G(\lambda_{k_1},\mu_{k_2}) - \sum_{b,a=1}^{n-l} F_a(\mu_{k_2})\left[D^{-1}\right]_{ab}F_b(\lambda_{k_1})\right] ,\end{aligned} \tag{C.32}$$

Since the computation of D^{-1} is challenging, we express the matrix kernel in the determinant above as an eigenvalue integral, see Ref. [111]. Considering Eq. (5.15) with m ($m \leq n$) instead of n integrals as well as 2 instead of p empirical eigenvalues, leads to

$$\begin{aligned} Z_m^{0/2}(\Lambda_1, \Lambda_2) &= \int \frac{\mathrm{d}[Y]\ \Delta_m^2(Y) \exp\left(\mathrm{tr}\left(\imath Y + \mathbf{1}_m\right)\right)}{\det^n(\imath Y + \mathbf{1}_m)^n \prod_{k=1}^2 \det\left(t/\Lambda_k + \imath Y + \mathbf{1}_m\right)} \\ =& \frac{(2\pi)^{m-1} m! (-1)^{m+1}}{(m-1)!} \prod_{j=0}^{m-2} \frac{(j+1)!}{(n-j-1)!} \\ \times& \left(G(\Lambda_1, \Lambda_2) - \sum_{b,a=1}^{m-1} F_a(\Lambda_1) \left[D^{-1}\right]_{ab} F_b(\Lambda_2) \right) \end{aligned} \tag{C.33}$$

If we compare Eq. (C.32) with Eq. (C.33) for $m = n - l + 1$, we obtain that the kernel in the determinant of Eq. (C.32) is proportional to $Z_{n-l+1}^{0/2}$ such that

$$\begin{aligned} E_p^{(2)}([0,t];p) &= \frac{(-1)^{nl+l(l+1)/2} t^{np-l(l-1)} (2\pi)^{-l}}{\det^{n-l+1}\Lambda \Delta_l(\lambda)\Delta_l(\mu)(n-l)!} \prod_{j=0}^{n-l-1} \frac{(j+1)!}{(n-j-1)!} \\ \times \det& \left[\frac{(-1)^{n-l}(n-l)! \prod_{j=0}^{n-l-1}(n-j-1)! Z_{n-l+1}^{0/2}(\lambda_{k_1}, \mu_{k_2})}{(2\pi)^{n-l}(n-l+1)! \prod_{j=0}^{n-l-1}(j+1)!} \right] , \end{aligned} \tag{C.34}$$

where $1 \leq k_1, k_2 \leq l$.

C.5 Computation of the Two-Point Kernel (C.33)

Analogous to the second approach, we map Eq. (C.33) to a corresponding Wishart matrix model, see appendix C.11. We apply section 5.2.1 backwards. Diagonalizing the matrix model arising and using Eq. (C.1) for the resulting group integral, we find

$$\begin{aligned} & Z_{n-l+1}^{0/2}(\lambda_{k_1}, \mu_{k_2}) \\ & \sim \int_0^1 \mathrm{d}x \mathrm{d}y \frac{(x-y)(xy)^{n-l-1}(1-x)^{l-1}(1-y)^{l-1}}{t(\lambda_{k_1} - \mu_{k_2})} \exp\left(-\frac{tx}{\lambda_{k_1}} - \frac{ty}{\mu_{k_2}}\right) \end{aligned} \tag{C.35}$$

$$\sim \frac{\varphi_{n-l}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right) \varphi_{n-l-1}^{l-1}\left(\frac{t}{\mu_{k_2}}\right) - \varphi_{n-l-1}^{l-1}\left(\frac{t}{\lambda_{k_1}}\right) \varphi_{n-l}^{l-1}\left(\frac{t}{\mu_{k_2}}\right)}{t(\lambda_{k_1} - \mu_{k_2})} . \tag{C.36}$$

To compute the proportionality constant, we compare Eq. (C.33) with Eq. (C.36) in a rescaled $t \to \infty$ limit. For the initial matrix model, we obtain

$$\lim_{t\to\infty} \frac{t^{2(n-l+1)} Z_{n-l+1}^{0/2}(\lambda_{k_1}, \mu_{k_2})}{(\lambda_{k_1}\mu_{k_2})^{n-l+1}} = (2\pi)^{n-l+1} \prod_{j=0}^{n-l} \frac{(j+1)!}{(n-1-j)!} , \tag{C.37}$$

whereas the same limit applied to Eq. (C.36) leads to

$$\lim_{t\to\infty} \frac{t^{2(n-l+1)} Z^{0/2}_{n-l+1}(\lambda_{k_1}, \mu_{k_2})}{(\lambda_{k_1}\mu_{k_2})^{n-l+1}} = K' \frac{(n-l)!(n-l-1)!}{\lambda_{k_1}\mu_{k_2}} , \tag{C.38}$$

where K' is the desired proportionality constant.

C.6 Derivation of Expression (5.41)

In this section, we apply the results of Ref. [112] to derive the expression (5.41). First, we write the Vandermonde determinant to a power of four in Eq. (5.39) as

$$\Delta_n^4(Y) = \int\limits_{-\infty}^{\infty} \mathrm{d}y_{n+1}\cdots \mathrm{d}y_{2n}\Delta_{2n}(y_1,\ldots,y_{2n}) \prod_{i=1}^{n} \frac{\delta(y_i - y_{i+n})}{(y_i - y_{i+n})} , \tag{C.39}$$

and therefore extend $Y = \mathrm{diag}(y_1,\ldots,y_n)$ to $\bar{Y} = \mathrm{diag}(y_1,\ldots,y_{2n})$. Because of the n additional integrals and the insertion of the δ-functions, the power of two of the determinants in the denominator of Eq. (5.39) reduces to a power of one. We use the translation invariance of the Vandermonde determinant and cast the resulting integrand into the form

$$\begin{aligned}
&\frac{\prod_{j=1}^{n} \exp\left(2(\imath y_j + 1)\right)}{\prod_{j=1}^{n} (\imath y_1 + \mathbf{1}_n)^{2n-1}} \frac{\Delta_{2n}(\bar{Y})}{\prod_{k,i=1}^{p,2n} (\imath y_i + 1 + 2t/\Lambda_k)} \prod_{i=1}^{n} \frac{\delta(y_i - y_{i+n})}{(y_i - y_{i+n})} \\
&= \frac{\imath^{n^2}(-1)^{n(n+1)/2+p(p+1)/2}}{(2t)^{p(p-1)/2}\mathrm{det}^{1-p}\Lambda\Delta_p(\Lambda)} \\
&\mathrm{det}\left[\begin{array}{c|c}
\dfrac{(\imath y_j + 1)^{1-2n}\exp\left(2(\imath y_j+1)\right)}{(\imath y_j + 1 + 2t/\Lambda_k)} & \dfrac{\delta\left(y_j - y_{j+n}\right)}{(y_j - y_{j+n})\left(\imath y_{j+n} + 1 + 2t/\Lambda_k\right)} \\
\hline
\dfrac{\exp 2(\imath y_j + 1)}{(\imath y_j + 1)^{2n-l}} & \dfrac{(\imath y_{j+n} + 1)^{l-1}\delta\left(y_j - y_{j+n}\right)}{(y_j - y_{j+n})}
\end{array}\right] ,
\end{aligned} \tag{C.40}$$

where $1 \leq j \leq n$, $1 \leq l \leq 2n-p$ and $1 \leq k \leq p$. Substituting Eq. (C.40) together with the n additional integrals (C.39) into the gap probability (5.39), yields

$$\begin{aligned}
E_p^{(4)}([0,t];p) &= \frac{K\imath^{n^2}(-1)^{n(n+1)/2+p(p+1)/2}t^{2np-p(p-1)/2}}{2^{p(p-1)/2}\mathrm{det}^{1-p}\Lambda\Delta_p(\Lambda)} \\
&\int \mathrm{d}[Y]\ \mathrm{det}\left[\begin{array}{c|c}
\dfrac{(\imath y_j + 1)^{1-2n}\exp\left(2(\imath y_j+1)\right)}{(\imath y_j + 1 + 2t/\Lambda_k)} & \displaystyle\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}z\delta\left(y_j - z\right)}{(y_j - z)\left(\imath z + 1 + 2t/\Lambda_k\right)} \\
\hline
\dfrac{\exp\left(2(\imath y_j + 1)\right)}{(\imath y_j + 1)^{2n-l}} & \displaystyle\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}z(\imath z + 1)^{l-1}\delta\left(y_j - z\right)}{(y_j - z)}
\end{array}\right] .
\end{aligned} \tag{C.41}$$

The matrix in the determinant above separates into two blocks. Both are $2n \times n$ dimensional and in each block the columns depend on one eigenvalue y_i with $i = 1, \ldots, n$ only. Accordingly, we apply de Bruijn's integral theorem (*c.f.* appendix of Ref. [111]), leading to a Pfaffian determinant of dimension $2n \times 2n$ such that

$$E_p^{(4)}([0,t];p) = \frac{K\imath^{n^2}(-1)^{n+p(p+1)/2}n!t^{2np-p(p-1)/2}}{2^{p(p-1)/2}\det^{1-l}\Lambda\Delta_p(\Lambda)}\mathrm{pf}\begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}. \tag{C.42}$$

The entries of A, B and C in the Pfaffian, consist of double integrals. Because of a delta function, one of these integrals is trivial such that the kernels are effectively given by

$$A_{kk'} = -\imath\frac{2t\,(\Lambda_k - \Lambda_{k'})}{\Lambda_{k'}\Lambda_k}2^{2n-6}\oint\limits_{\mathcal{C}_4}\frac{\mathrm{d}w\exp(w)}{w^{2n-1}\,(w+t/\Lambda_{k'})^2\,(w+t/\Lambda_k)^2}\ , \tag{C.43}$$

$$B_{lk} = (-\imath)^2 2^{2n-l-1}\oint\limits_{\mathcal{C}_4}\frac{\mathrm{d}w\,(t(l-1)/\Lambda_k + lw)\exp(w)}{w^{2n+1-l}\,(w+t/\Lambda_k)^2}\ . \tag{C.44}$$

and

$$C_{ll'} = 2^{2n-l-l'-1}(-\imath)^2(l-l')\oint\limits_{\mathcal{C}_4}\frac{\mathrm{d}w\exp(w)}{w^{2n+2-l'-l}}\ . \tag{C.45}$$

Previously, we have introduced the contour integrals by the same arguments as in the case of the complex ensemble, see also Eq. (C.9). The exponent ensures convergence when closing the contour over the real line by an arc in the complex upper half-plane starting and ending at plus and minus infinity. We find a contour that encloses all poles of the integrand, respectively. To compute the resulting path integral we apply the residue theorem, leading to

$$\begin{aligned}
\frac{2^{-2n+6}\Lambda_{k'}\Lambda_k}{(-\imath)2t\,(\Lambda_k-\Lambda_{k'})}A_{kk'} = {}& \frac{2\pi\imath\Lambda_{k'}^2\Lambda_k^2\ e^{-t/\Lambda_k}}{(\Lambda_k-\Lambda_{k'})^2}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-1}\right.\\
&+\left(-\frac{\Lambda_{k'}}{t}\right)^{2n-1} - (2n-1)\left(-\frac{\Lambda_k}{t}\right)^{2n} - (2n-1)\left(-\frac{\Lambda_{k'}}{t}\right)^{2n}\\
&\left.+\sum_{m=0}^{2n-2}\frac{(-1)^m(m+1)\ e^{t/\Lambda_k}}{(2n-2-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2}+\left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right]\\
&\frac{4\pi\imath\Lambda_{k'}^3\Lambda_k^3\ e^{-t/\Lambda_k}}{(\Lambda_k-\Lambda_{k'})^2}\left[\left(-\frac{\Lambda_k}{t}\right)^{2n-1}+\left(-\frac{\Lambda_{k'}}{t}\right)^{2n-1}\right.\\
&\left.+\sum_{m=0}^{2n-2}\frac{(-1)^m\ e^{t/\Lambda_k}}{(2n-2-m)!}\left(\left(\frac{\Lambda_k}{t}\right)^{m+2}+\left(\frac{\Lambda_{k'}}{t}\right)^{m+2}\right)\right]\ ,
\end{aligned} \tag{C.46}$$

for the upper left block,

$$
\begin{aligned}
\frac{B_{lk}}{(-\imath)^2 2^{2n-l-1}} &= \frac{(l-1)t}{\Lambda_k} 2\pi\imath\, e^{-t/\Lambda_k} \Bigg[\left(-\frac{\Lambda_k}{t}\right)^{2n-l} - (2n-l)\left(-\frac{\Lambda_k}{t}\right)^{2n-l+1} \\
&+ \sum_{m=0}^{2n-l} \frac{(-1)^m (m+1)\, e^{t/\Lambda_k}}{(2n-l-m)!} \left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2} \right) \Bigg] \\
&+ l 2\pi\imath\, e^{-t/\Lambda_k} \Bigg[\left(-\frac{\Lambda_k}{t}\right)^{2n-l-1} - (2n-l-1)\left(-\frac{\Lambda_k}{t}\right)^{2n-1} \\
&+ \sum_{m=0}^{2n-l-1} \frac{(-1)^m (m+1)\, e^{t/\Lambda_k}}{(2n-l-1-m)!} \left(\left(\frac{\Lambda_k}{t}\right)^{m+2} + \left(\frac{\Lambda_{k'}}{t}\right)^{m+2} \right) \Bigg]
\end{aligned} \tag{C.47}
$$

for the off diagonal block and

$$
\frac{2^{+l+l'+1-2n} C_{ll'}}{(-\imath)^2 (l-l')} = \begin{cases} \dfrac{2\pi\imath}{(2n-l-l'+1)!} & , 2n+1 \geq l+l' \\ 0 & , 2n+1 < l+l' \end{cases} \tag{C.48}
$$

for the lower right block.

C.7 Derivation of Expression (5.46)

We give a detailed derivation of Eq. (5.46). With the aid of the weight (5.45), we cast the integrand in Eq. (5.39) again into a determinantal form, but in a different way such that

$$
\begin{aligned}
E_p^{(4)}([0,t];p) &= \frac{K\imath^{n^2}(-1)^{n(n+1)/2+p(p+1)/2} t^{2np}}{(2t)^{p(p-1)/2} \det^{1-p}\Lambda \Delta_p(\Lambda)} \\
&\times \int d[Y] \det \left[\begin{array}{c|c} \dfrac{1}{\imath y_j + 1 + 2t/\Lambda_k} & \displaystyle\int_{-\infty}^{\infty} \frac{dz g(z, y_j)}{\imath z + 1 + 2t/\Lambda_k^{-1}} \\ \hline (\imath y_j + 1)^{l-1} & \displaystyle\int_{-\infty}^{\infty} dz g(z, y_j)(\imath z + 1)^{l-1} \end{array} \right] .
\end{aligned} \tag{C.49}
$$

The determinant in the second line of Eq. (C.49) has a similar structure to the one obtained in Eq. (C.41). Employing the similar arguments, we apply de Bruijn's integral theorem (*c.f.* appendix of Ref. [111]), leading to a $2n \times 2n$ dimensional Pfaffian structure given by

$$
E_p^{(4)}([0,t];p) = \frac{K\imath^{n^2}(-1)^{n+p(p+1)/2} n! t^{2np}}{(2t)^{p(p-1)/2} \det^{1-p}\Lambda \Delta_p(\Lambda)} \mathrm{pf} \begin{bmatrix} \mathcal{G}(\Lambda_k, \Lambda_{k'}) & \mathcal{F}_j(\Lambda_k) \\ -\mathcal{F}_i(\Lambda_{k'}) & (M_d)_{ij} \end{bmatrix} , \tag{C.50}
$$

where $1 \leq k, k' \leq p$, $1 \leq i, j \leq 2n-p$ and $d = 2n-p$. For clarity and comprehensibility we introduce

$$D_k^{(1)}(y_i) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z g(z, y_i)}{(\imath z + 1 + 2t/\Lambda_k)} \quad , \quad D_l^{(2)}(y_i) = \int_{-\infty}^{\infty} \mathrm{d}z g(z, y_j)(\imath z + 1)^{l-1}, \tag{C.51}$$

and

$$(M_d)_{ij} = \int_{-\infty}^{\infty} \mathrm{d}w \left((\imath w + 1)^{i-1} D_j^{(2)}(w) - (\imath w + 1)^{j-1} D_i^{(2)}(w) \right) , \tag{C.52}$$

$$\mathcal{G}\left(\Lambda_k, \Lambda_{k'}\right) = \int_{-\infty}^{\infty} \mathrm{d}w \left(\frac{D_{k'}^{(1)}(w)}{(\imath w + 1 + 2t/\Lambda_k)} - \frac{D_k^{(1)}(w)}{(\imath w + 1 + 2t/\Lambda_{k'})} \right) , \tag{C.53}$$

$$\mathcal{F}_j\left(\Lambda_k\right) = \int_{-\infty}^{\infty} \mathrm{d}w \left(\frac{D_j^{(2)}(w)}{(\imath w + 1 + 2t/\Lambda_k)} - (\imath w + 1)^{j-1} D_k^{(1)}(w) \right) , \tag{C.54}$$

The M_d is $d \times d$ dimensional, where $d = 2n - p = 2n - 2l$. It is even dimensional if p is even only. For technical reasons, we assume that $p = 2l$ is even and derive an expression for odd p from the results found for even p later on.

Similar to the complex case, we use the Schur complement to decrease the dimension in the Pfaffian. Let C be an even dimensional, invertible and skew-symmetric matrix, then

$$\mathrm{pf} \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix} = \mathrm{pf} C \ \mathrm{pf}\left[A + BC^{-1}B^T \right] . \tag{C.55}$$

Because d is even, the matrix M_d is even dimensional such that we can apply Eq. (C.55) to the Pfaffian determinant in Eq. (C.50), yielding

$$\begin{aligned} E_p^{(4)}([0,t];p) = {} & \frac{K \imath^{n^2} (-1)^{n+p(p+1)/2} n! \ t^{2np}}{(2t)^{p(p-1)/2} \det^{1-p}\Lambda \Delta_p(\Lambda)} \mathrm{pf} M_d \\ & \times \mathrm{pf} \left[\mathcal{G}\left(\Lambda_k, \Lambda_{k'}\right) + \sum_{i,j=1}^{d} \mathcal{F}_j\left(\Lambda_k\right) (M_d^{-1})_{ji} \mathcal{F}_i\left(\Lambda_{k'}\right) \right] , \end{aligned} \tag{C.56}$$

where $1 \leq k, k', \leq 2l$. The Pfaffian of M_d is readily derived from Eq. (C.50). By setting the number of empirical eigenvalues p to zero, the number of eigenvalue integrals n to $d/2$ and keeping the exponent of $(\imath z + 1)$ in the weight (5.45) to be $2n - 1$, we find

$$\mathrm{pf} M_d = (2\pi)^{d/2} \frac{\imath^{(d/2)^2} (-1)^{d/2}}{(d/2)!} 2^{d(2n-d-1)/2} \prod_{j=0}^{d/2-1} \frac{(2j+2)!}{(2n-2-2j)!} . \tag{C.57}$$

If we compute the remaining integrals and invert M_d we are finished. Unfortunately, we have not found a method to invert M_d. To provide a remedy, we compare the kernel of the Pfaffian (C.56) with the eigenvalue integral

$$Z_N^{0/2}(\Lambda_k,\Lambda_{k'}) = \int \frac{\mathrm{d}[Y]\ |\Delta_N(Y)|^4 \det^{1-2n}(\imath Y+\mathbf{1}_N)\exp(2\mathrm{tr}(\imath Y+\mathbf{1}_N))}{\prod_{i=k,k'}\det^2(2t/\Lambda_i\mathbf{1}_N+\imath Y+\mathbf{1}_N)} . \tag{C.58}$$

This eigenvalue integral is up to an overall normalization constant, the monomial prefactor t the integral (5.39) with n, p replaced by N, respectively, two. To ensure convergence, we assume that $N\leq n$. It has two empirical eigenvalues only and is readily solved using the results found earlier,

$$\begin{aligned}&\frac{(-\imath)^{-N^2-(N-1)^2}t(\Lambda_k-\Lambda_{k'})\prod_{j=0}^{N-2}(2n-2-2j)!}{(2\pi)^{N-1}2^{N-2}N\Lambda_k\Lambda_{k'}\prod_{j=0}^{N-2}(2j+2)!}Z_N^{0/2}(\Lambda_k,\Lambda_{k'})\\ &=\left(\mathcal{G}(\Lambda_k,\Lambda_{k'})+\sum_{i,j=1}^{2N-2}\mathcal{F}_j(\Lambda_k)(M_{2N-2}^{-1})_{ji}\mathcal{F}_i(\Lambda_{k'})\right) .\end{aligned} \tag{C.59}$$

Hence, if we set $N=(d+2)/2=n+1-l$, the matrix kernel in the Pfaffian (C.56) is proportional to the eigenvalue integral (C.59).

C.8 Derivation of the Matrix Kernel (5.48)

To express Eq. (C.58) by a full real-quaternion self-dual matrix model, we consider

$$\begin{aligned}Z_{n+1-l}^{0/2}(\Lambda_k,\Lambda_{k'}) &= K_{n+1-l}\int \mathrm{d}[H]\ \frac{\exp\left(\mathrm{tr}\left(\imath H+\mathbf{1}_{2(n+1-l)}\right)\right)}{\det^{(n-1/2)}\left(\imath H+\mathbf{1}_{2(n+1-l)}\right)}\\ &\times\prod_{i=k,k'}\det{}^{-1}\left(2t/\Lambda_i\mathbf{1}_{2(n+1-l)}+\imath H+\mathbf{1}_{2(n+1-l)}\right) .\end{aligned} \tag{C.60}$$

If we diagonalize H, we are up to an overall constant left with Eq. (C.58). The normalization constant K_{n+1-l} is determined at the end of the calculation by a comparison with Eq. (C.58). To rewrite the determinant in the denominator, we introduce a Gaussian integral over a real quaternion Wishart matrix B of dimension $4\times 2(n+1-l)$. Exchanging the H and the B integral and using Eq. (D.4) to perform the former, yields

$$\begin{aligned}Z_{n+1-l}^{0/2}(\Lambda_k,\Lambda_{k'}) &= K_{n+1-l}\int \mathrm{d}[B]\ P(B|t\tilde{\Lambda})\\ &\times\Theta(\mathbf{1}_{2(n+1-l)}-B^\dagger B)\det{}^{2l-2}\left(\mathbf{1}_{2(n+1-l)}-B^\dagger B\right) ,\end{aligned} \tag{C.61}$$

where $\tilde{\Lambda}=\mathrm{diag}(\Lambda_k\mathbf{1}_2,\Lambda_{k'}\mathbf{1}_2)$. We use the invariance of the Θ-function and the determinant under $BB^\dagger\longleftrightarrow B^\dagger B$, as explained below Eq. (5.5) and replace in Eq. (C.61) the large matrix $B^\dagger B$ by the small 4×4 matrix $BB^\dagger$. Thus, we break down the computation of the kernel (C.58) to a 4×4 real-quaternion self-dual matrix integral. The latter possesses the same symmetry breaking exponent, present at the beginning in Eq. (5.5). The difference is that the resulting Itzykson-Zuber integral is known. Hence, we diagonalize $BB^\dagger = U(X\otimes \mathbf{1}_2)U^\dagger$ with $X = \mathrm{diag}(x,y)$, $U \in$ USp(4) and obtain

$$
\begin{aligned}
Z^{0/2}_{n+1-l}\left(\Lambda_k,\Lambda_{k'}\right) = K_{n+1-l}\int\limits_0^1 \mathrm{d}x\mathrm{d}y(x-y)^4(xy)^{2n-2l-1} \\
\times (1-x)^{2l-2}(1-y)^{2l-2}\Phi_4(X\otimes\mathbf{1}_2|t\tilde{\Lambda}^{-1})\ ,
\end{aligned}
\tag{C.62}
$$

where

$$
\Phi_4(X\otimes\mathbf{1}_2|t\tilde{\Lambda}^{-1}) = \int \mathrm{d}\mu(U)\exp\left(-2t\mathrm{tr}U\tilde{X}U^\dagger\tilde{\Lambda}^{-1}\otimes\mathbf{1}_2\right)\ , \tag{C.63}
$$

The integral (C.63) is over USp(4) with respect to the Haar measure $\mathrm{d}\mu(U)$. It is the first, non-trivial example of the unitary-symplectic Itzykson-Zuber integral and is computed in Ref. [67]. It arises in appendix A.8 when discussing the alternative approach, see Eq. (A.55). Inserting the expression derived there into the twofold eigenvalue integral (C.62), we arrive at

$$
\begin{aligned}
&Z^{0/2}_{n+1-l}\left(\Lambda_k,\Lambda_{k'}\right) = K_{n+1-l}\int\limits_0^1 \mathrm{d}x\mathrm{d}y(x-y)^4(xy)^{2n-2l-1}(1-x)^{2l-2}(1-y)^{2l-2} \\
&\times\left(-\frac{\exp\left(-2tx\Lambda_{k'}^{-1}-2ty\Lambda_k^{-1})\right)}{\left(t(x-y)(\Lambda_k^{-1}-\Lambda_{k'}^{-1})\right)^3}+\frac{\exp\left(-2tx\Lambda_{k'}^{-1}-2ty\Lambda_k^{-1})\right)}{\left(t(x-y)(\Lambda_k^{-1}-\Lambda_{k'}^{-1})\right)^2}\right)
\end{aligned}
\tag{C.64}
$$

The remaining integrals are simple and we express them in terms of $\varphi^s_m(x)$ introduced in Eq. (5.22). To this end, we expand the Vandermonde determinant and find

$$
\begin{aligned}
&\frac{Z^{0/2}_{n+1-l}\left(\Lambda_k,\Lambda_{k'}\right)t^2\left(\Lambda_k-\Lambda_{k'}\right)^2}{(\Lambda_k\Lambda_{k'})^2K_{n+1-l}} = -2\varphi^{2l-2}_{2n-2l}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi^{2l-2}_{2n-2l}\left(\frac{2t}{\Lambda_k}\right) \\
&+\varphi^{2l-2}_{2n-2l+1}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi^{2l-2}_{2n-2l-1}\left(\frac{2t}{\Lambda_k}\right)+\varphi^{2l-2}_{2n-2l-1}\left(\frac{2t}{\Lambda_{k'}}\right)\varphi^{2l-2}_{2n-2l+1}\left(\frac{2t}{\Lambda_k}\right) \\
&-\frac{\varphi^{2l-2}_{2n-2l}\left(\frac{2t}{\Lambda_k}\right)\varphi^{2l-2}_{2n-2l-1}\left(\frac{2t}{\Lambda_{k'}}\right)-\varphi^{2l-2}_{2n-2l-1}\left(\frac{2t}{\Lambda_k}\right)\varphi^{2l-2}_{2n-2l}\left(\frac{2t}{\Lambda_{k'}}\right)}{t(\Lambda_k\Lambda_{k'})^{-1}\left(\Lambda_k-\Lambda_{k'}\right)} \\
&=\frac{t\left(\Lambda_k-\Lambda_{k'}\right)}{\left(\Lambda_k\Lambda_{k'}\right)}\,\Xi\left(\Lambda_k,\Lambda_{k'}\right)\ .
\end{aligned}
\tag{C.65}
$$

The normalization constant K_{n+1-l} is determined by comparing Eq. (C.58) with Eq. (C.65) in the limit $t\to\infty$. This limit is meaningful only if we rescale both expressions with $t^{4n+4-2l}$. We use the asymptotic expansion (5.25) of φ_m^s to compute the limit of Z_{n+1-l} and find

$$\lim_{t\to\infty} Z_{n+1-l}^{0/2}(\Lambda_k,\Lambda_{k'}) = K_{n+1-l}\left(\frac{\Lambda_k\Lambda_{k'}}{4}\right)^{2n-2l+2}(2n-2l-1)!(2n-2l+1)! \tag{C.66}$$

whereas the same limit applied to Eq. (C.58) yields

$$\begin{aligned}\lim_{t\to\infty} Z_{n+1-l}^{0/2}(\Lambda_k,\Lambda_{k'}) = &\left(\frac{1}{2}\right)^{(2n+1-2l)(n+2-l)}(2\pi)^{n-l+1}\\ &\left(\frac{\Lambda_k\Lambda_{k'}}{4}\right)^{2n-2l+2}\prod_{j=0}^{n-l}\frac{(2j+2)!}{(2n-2-2j)!}\end{aligned} \tag{C.67}$$

such that the normalization constant is given by

$$K_{n+1-l} = \frac{(1/2)^{(2n+1-2l)(n+2-l)}\,(2\pi)^{n-l+1}}{(2n-2l-1)!(2n-2l+1)!}\prod_{j=0}^{n-l}\frac{(2j+2)!}{(2n-2-2j)!}\ . \tag{C.68}$$

C.9 Derivation of the Pfaffian Expression (5.60)

We devote this section to a detailed derivation of the Pfaffian expression (5.60). The normalization constant in Eq. (5.58) is determined by the condition that $E_{2p}^{(1)} \to 1$ for $t\to\infty$, yielding

$$K = \frac{2^{2n(p-2)}}{\det^{2n}\Lambda\ n!\pi^n}\ . \tag{C.69}$$

Like for the complex and the real-quaternion case, averages of products of characteristic polynomials in real symmetric ensembles with respect to an arbitrary weight $w(x)$ were studied extensively in Refs. [107, 110, 112, 173] and references therein. We use the results of Ref. [112] and write

$$\begin{aligned}\frac{\prod_{i<j}(\imath y_i-\imath y_j)}{\prod_{i=1}^{2n}\prod_{k=1}^{p}(\dfrac{t}{2\Lambda_k}+1+\imath y_i)} &= (-1)^{p(p-1)/2+2n(2n-1)/2-p(2n+1)}\\ \times\frac{\imath^{2n(2n-1)/2+(2n-p)(2n-p-1)/2}}{\det^{1-p}\Lambda\Delta_p(\Lambda)}&\det\left[\begin{array}{c}\dfrac{1}{\left(\dfrac{t}{2\Lambda_k}+1+\imath y_i\right)}\\ y_i^{j-1}\end{array}\right]\ ,\end{aligned} \tag{C.70}$$

where $1\le k\le p$, $1\le i\le 2n$ and $1\le j\le 2n-p$. Substituting Eq. (C.70) into the eigenvalue integral (5.58), symmetrizing the latter and applying the method of alternating variables, we obtain

$$E_p^{(2)}([0,t];p) = K_1 K t^{np2} \frac{(2n)!(-1)^{n(n-1)/2}}{n!} \int \mathrm{d}y_2 \ldots \mathrm{d}y_{2n}$$
$$\times \det \left[\begin{array}{c|c} \int\limits_{-\infty}^{y_{2i}} \frac{\mathrm{d}x w_1(x)}{\left(\frac{t}{2}\Lambda_k^{-1} + 1 + \imath x\right)} & \frac{w(y_{2i})}{\left(\frac{t}{2}\Lambda_k^{-1} + 1 + \imath y_{2i}\right)} \\ \hline \int\limits_{-\infty}^{y_{2i}} \mathrm{d}x w_1(x) x^{j-1} & w(y_{2i}) y_{2i}^{j-1} \end{array} \right] , \tag{C.71}$$

where K_1 is the prefactor in front of the determinant on the right hand side of Eq. (C.70). The determinant is of dimension $2n \times 2n$ and separates into two blocks of dimension $n \times p$ and $n \times (2n-p)$. Within each of these blocks, the columns depend on one of the integration variables only. Hence, the integral is appropriate to apply de Bruijn's integral theorem (c.f. Ref. [112]), leading to an expression with a $2n \times 2n$-dimensional Pfaffian determinant

$$E_p^{(1)}([0,t];p) = K_1 K t^{np2} (2n)! \mathrm{pf} \left[\begin{array}{c|c} G^{(1)}(\Lambda_k, \Lambda_{k'}) & \mathcal{F}_j^{(1)}(\Lambda_k) \\ \hline -\mathcal{F}_i^{(1)}(\Lambda_{k'}) & (M_{2n-p})_{ij} \end{array} \right] . \tag{C.72}$$

For the sake of clarity and comprehensibility, we introduce

$$D_{1,k}^{(1)}(y_{2i}) = \int\limits_{-\infty}^{y_{2i}} \frac{\mathrm{d}x w_1(x)}{(t/(\Lambda_k 2) + 1 + \imath x)} \quad ; \quad D_{1,k}^{(2)}(y_{2i}) = \int\limits_{-\infty}^{y_{2i}} \mathrm{d}x w_1(x) x^{j-1} \tag{C.73}$$

and

$$[M_{2n-p}]_{ij} = \int\limits_{-\infty}^{\infty} \mathrm{d}x \left(w(x) x^{i-1} D_{1,j}^{(2)}(x) - w(x) x^{j-1} D_{1,i}^{(2)}(x) \right) , \tag{C.74}$$

$$\mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'}) = \int\limits_{-\infty}^{\infty} \mathrm{d}x \left(\frac{w(x) D_{k'}^{(1)}(x)}{(\imath x + 1 + t/(\Lambda_k 2))} - \frac{w(x) D_{1,k}^{(1)}(x)}{(\imath x + 1 + t/(\Lambda_{k'} 2))} \right) , \tag{C.75}$$

$$\mathcal{F}_j^{(1)}(\Lambda_k) = \int\limits_{-\infty}^{\infty} \mathrm{d}x \left(\frac{w(x) D_{1,j}^{(2)}(x)}{(\imath x + 1 + t/(\Lambda_k 2))} - x^{j-1} w(x) D_{1,k}^{(1)}(x) \right) , \tag{C.76}$$

where $1 \le k, k' \le p$, and $1 \le i, j \le 2n - p$. Thus, we reduce the computation of the $2n$ coupled integrals in Eq. (5.58) to the calculation of a coupled twofold integral. This is due to the coupling and is challenging. It is unclear how to express it in terms of known functions.

It is important to emphasize that the Pfaffian structure shown above, is not known in the literature. Although we assume a doubly degenerate empirical eigenvalue spectrum, the Pfaffian structure can neither be anticipated from Eq. (2.36) nor from Eq. (5.5). It is still $2n \times 2n$ dimensional. To obtain a smaller matrix kernel, we assume for technical reasons that $p = 2l$ is even. As in the previous sections, we derive the odd p case as a special limit.

The matrix M_{2n-p} is $(2n-p) \times (2n-p)$ dimensional. If p is even, M_{2n-p} is even dimensional. It is invertible, because its Pfaffian is non-vanishing. To observe this, take the number of eigenvalue integrals in Eq. (5.58) to be $2n-p$ instead of n and set all eigenvalues to zero. Applying the same techniques we use to obtain Eq. (C.72), yields

$$\mathrm{pf} M_{2n-2l} = \frac{(-1)^{n-l}(n-l)!\, 4^{2n-2l}}{(2n-2l)!} \prod_{j=0}^{2n-2l-1} \frac{\Gamma\left((j+3)/2\right)}{\Gamma\left((2n+1-j)/2\right)} . \tag{C.77}$$

Thus, since M_{2n-p} is even dimensional and invertible, we can apply the Schur complement (C.55) and arrive at

$$\begin{aligned} E_p^{(1)}([0,t];p) &= K_1 K t^{np2} (2n)! \mathrm{pf} M_{2n-2l} \\ &\times \mathrm{pf}\left[\mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'}) + \sum_{i,j=1}^{2n-2l} \mathcal{F}_j^{(1)}(\Lambda_k)(M_{2n-2l}^{-1})_{ji} \mathcal{F}_i^{(1)}(\Lambda_{k'})\right] , \end{aligned} \tag{C.78}$$

where $1 \le k, k' \le 2l$.

C.10 Computation of $\mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'})$

The difficulty in the computation is caused by the coupling of two integrals (5.64). To decouple the integrals, we introduce a Heaviside Θ-function. We express it in terms of its integral representation such that we arrive at

$$\begin{aligned} \mathcal{G}^{(1)}(\Lambda_k, \Lambda_{k'}) = &\frac{(-\imath)}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \mathrm{d}y \mathrm{d}\tau \frac{\exp\left((1-\tau^-)(\imath y_i + 1) + (1+\tau^-)(\imath x_i + 1)\right)}{\tau^-(\imath y_i + 1)^\alpha (\imath x_i + 1)^\alpha} \\ \times &\left(\frac{1}{(\imath x + 1 + t/(\Lambda_k 2))(\imath y + 1 + t/(\Lambda_{k'} 2))} - (x \leftrightarrow y)\right) , \end{aligned} \tag{C.79}$$

where $\tau^- = \tau - \imath$ and $\alpha = (2n+1)/2$. According to the τ integral, the x and the y integral decouple. The resulting integrals are of the same kind and can be done using Eq. (D.4), yielding for the x integral

$$\begin{aligned} &\int_{-\infty}^{\infty} \mathrm{d}x \frac{\exp\left((1+\tau^-)(\imath x_i + 1)\right)}{(\imath x_i + 1)^\alpha (\imath x + 1 + t/(\Lambda_k 2))} \\ =& \frac{2\pi(\tau^- + 1)^\alpha \Theta(\tau + 1)}{\Gamma(\alpha)} \int_0^1 \mathrm{d}u (1-u)^{\alpha-1} \exp\left(-u \frac{t(\tau^- + 1)}{2\Lambda_k}\right) \\ =& \frac{2\pi(\tau^- + 1)^\alpha \Theta(\tau + 1)}{\Gamma(\alpha)} \varphi_0^{\alpha-1}\left(\frac{t(\tau^- + 1)}{2\Lambda_k}\right) , \end{aligned} \tag{C.80}$$

where φ_s^m was introduced in Eq. (5.22). For $m \in \mathbb{N}$ the function $\varphi_s^{m/2}$ is not a polynomial but can be expressed as

$$\varphi_s^{m/2}(x) = \frac{(-\imath)^{m+2}\exp(-x)}{x^{m/2-1}}\left(\Gamma\left(\frac{m+2}{2}\right) - \Gamma\left(\frac{m+2}{2};-x\right)\right) \ , \tag{C.81}$$

where $\Gamma(z;-x)$ is the incomplete Γ-function. Analogously, we can express the y integral in Eq. (C.79) in terms of φ_s^m with $(\tau^- + 1)$ replaced by $(1-\tau^-)$. What follows is that the coupled twofold integral reduces to a single integral

$$\begin{aligned}\mathcal{G}^{(1)}\left(\Lambda_k, \Lambda_{k'}\right) = &\frac{(-\imath)}{\Gamma^2((2n+1)/2)}\int\limits_{-1}^{1} \mathrm{d}\tau \frac{(1-(\tau^-)^2)^{n-1/2}}{\tau-\imath} \\ &\times\left(\varphi_0^{n-1/2}\left(\frac{t(\tau^-+1)}{2\Lambda_k}\right)\varphi_0^{n-1/2}\left(\frac{t(1-\tau^-)}{2\Lambda_{k'}}\right) - (\Lambda_k \leftrightarrow \Lambda_{k'})\right) \ ,\end{aligned} \tag{C.82}$$

over a compact interval.

C.11 Cauchy Transform of Orthogonal Polynomials

The Cauchy transform can be reduced to a simple expression by employing the results of Ref. [174], where the authors show under general assumptions that

$$t_{2j}(x) = -\frac{r_j \imath^{2j+1}}{Z_{2j+2}^{0/0}}\int \mathrm{d}[Y] \ \frac{|\Delta_{2j+2}(Y)|\prod_{i=1}^{2j+2} w(y_i)}{\prod_{i=1}^{2j+2}(x+\imath y_i+1)} \tag{C.83}$$

for the Cauchy transform of polynomials of even degree and

$$t_{2j+1}(x) = -\frac{r_j \imath^{2j+1}}{Z_{2j+2}^{0/0}}\int \mathrm{d}[Y] \ \frac{|\Delta_{2j+2}(Y)|\left(\mathrm{tr}Y - \imath\left(x+1\right)+c_j\right)\prod_{i=1}^{2j+2} w(y_i)}{\prod_{i=1}^{2j+2}(x+\imath y_i+1)} \tag{C.84}$$

for the Cauchy transform of polynomials of odd degree. The overall minus sign in Eqs. (C.83) and (C.84) can be omitted because it cancels in the bilinear form (5.79) and c_j is a free parameter yet to be determined. For the weight function (5.59), it is possible to relate t_{2j+1} and t_{2j}. The latter derives from the former by

$$t_{2j+1}(x) = -\imath\left(x + \frac{\partial}{\partial\eta}\right) t_{2j}(\eta x) \ , \tag{C.85}$$

where we fix c_j for convenience. To compute Eq. (C.85), we make use of the fact that the integral (C.84) is independent of shifting Y by an imaginary increment as long as no pole of the integrand is crossed.

To solve Eq. (C.83), we apply section 5.2.1 backwards and obtain an uncorrelated Wishart matrix model

$$\begin{aligned}t_{2j}(x) = K &\int \mathrm{d}[B] \ \exp\left(-x\mathrm{tr}BB^\dagger\right) \\ &\times \det{}^{n-j-1}\left(BB^\dagger - \mathbf{1}_2\right)\Theta\left(BB^\dagger - \mathbf{1}_2\right) \ ,\end{aligned} \tag{C.86}$$

where B is a $2 \times (2j+2)$ dimensional matrix with real entries. The normalization constant is read from $t_{2j}(x)$ and Eq. (C.86) in the limit $x \to \infty$ when both are rescaled by x^{2j+2}, leading to

$$K = -\frac{r_j \imath^{2j+1}}{\pi^{2j+2}} \ . \tag{C.87}$$

Since the Wishart model (C.86) is invariant, it depends only on the eigenvalues $X = \mathrm{diag}(x_1, x_2)$ of $BB^\dagger = UXU^\dagger$, where $U \in \mathrm{O}(2)$. The integral over $\mathrm{O}(2)$ yields a constant factor such that we are left with

$$\begin{aligned} t_{2j}(x) = -\frac{r_j \imath^{2j+1} 2^{2j-1}}{(2j)!} \int\limits_0^\infty & \mathrm{d}x_1 \mathrm{d}x_2 |x_1 - x_2| (x_1 x_2)^{(2j-1)/2} \\ &\times (1-x_1)^{n-j-1} (1-x_2)^{n-j-1} \exp\left(-x(x_1+x_2)\right) \ . \end{aligned} \tag{C.88}$$

At least one integral of the above can be performed by expanding $(1-x)^{n-j-1}$. Inserting Eq. (5.81) together with this expansion into the expression (C.88) yields

$$\begin{aligned} t_{2j}(x) = -&\frac{4\pi 2^{2n-2j} \imath^{2j+1}}{(2n-3-2j)!} \sum_{l_1,l_2=1}^{n-j-1} \begin{pmatrix} n-j-1 \\ l_1 \end{pmatrix} \begin{pmatrix} n-j-1 \\ l_2 \end{pmatrix} \\ &\times x^{l_1+l_2+2j+2} \left(\psi_{l_1+1,l_2}(x) - \psi_{l_1,l_2+1}(x) \right) \ , \end{aligned} \tag{C.89}$$

where

$$\begin{aligned} \psi_{\alpha_1,\alpha_2}(x) = &\int\limits_0^x \mathrm{d}x x^{(2j-1)/2+\alpha_2} \exp(-x) \\ &\times \left(\Gamma(\alpha_1 + (2j+1)/2) - \Gamma(\alpha_1 + (2j+1)/2; x) \right) \end{aligned} \tag{C.90}$$

For the remaining Cauchy transform t_{2j+1} of the odd degree polynomials, we make use of Eq. (C.85). The differentiation is straightforward and we obtain

$$\begin{aligned} t_{2j+1}(x) = &\frac{\imath t_{2j}(x)(x^2 + l_1 + l_2 + 2j + 2)}{x} \\ -&\frac{4\pi 2^{2n-2j} \imath^{2j}}{(2n-3-2j)!} \sum_{l_1,l_2=1}^{n-j-1} \begin{pmatrix} n-j-1 \\ l_1 \end{pmatrix} \begin{pmatrix} n-j-1 \\ l_2 \end{pmatrix} \\ +&x^{(2j+1)/2+l_2} \exp(-x) \left(\Gamma(l_1 + (2j+3)/2) - \Gamma(l_1 + (2j+3)/2; x) \right) \\ -& \, x^{(2j+3)/2+l_2} \exp(-x) \left(\Gamma(l_1 + (2j+1)/2) - \Gamma(l_1 + (2j+1)/2; x) \right) \Big] \ , \end{aligned} \tag{C.91}$$

where we make use of Eq. (C.89) to identify the first row of the expression above with the Cauchy transform of the even polynomials.

APPENDIX D

Ingham-Siegel Integral

We present a simple derivation of the solution to the Ingham-Siegel,

$$I_{\alpha,N}(A) = \int \mathrm{d}[H] \, \frac{\exp \mathrm{tr}(\imath H + \mathbf{1}_{\gamma_2 N})A}{\det^{\alpha/\gamma_1}(\imath H + \mathbf{1}_{\gamma_2 N})} \, , \tag{D.1}$$

where H and A are either real-symmetric, Hermitian or real-quaternion self-dual and $\alpha \geq N-1+2/\beta$. To compute the integral (D.1), we express the determinant in the denominator as Gaussian integral. After an exchange of the integrations, we are left with

$$\begin{aligned} I_{\alpha,N}(A) = C \exp(\mathrm{tr}A) \int \mathrm{d}[B] \, \det{}^{\beta(\alpha-N+1-2/\beta)/2} B \exp(-\mathrm{tr}B) \\ \times \int \mathrm{d}[H] \, \exp(\imath \mathrm{tr} H (A-B)) \, , \end{aligned} \tag{D.2}$$

where C is a normalization constant yet to be determined and B is a positive definite matrix in the same symmetry class as H and A. The resulting H integral is the matrix integral representation of the δ-function. Thus, if we insert this into Eq. (D.2), we find

$$I_{\alpha,N}(A) = C \int \mathrm{d}[B] \, \det{}^{\beta(\alpha-N+1-2/\beta)/2} B \delta(A-B) \exp(-\mathrm{tr}(B-A)) \, . \tag{D.3}$$

Since B is a positive definite matrix, the B integral is non-zero only if A is a positive definite matrix as well. Hence, we perform the B integral and find

$$I_{\alpha,N}(A) = C\Theta(A) \det{}^{\beta(\alpha-N+1-2/\beta)/2} A \, , \tag{D.4}$$

In order to derive the normalization constant C, we compare Eq. (D.1) and Eq. (D.4) for $A = \mathbf{1}_p$. For this particular choice of A Eq. (D.1) reduces to

$$\begin{aligned} \int \mathrm{d}[H] \, \frac{\exp \mathrm{tr}(\imath H + \mathbf{1}_{\gamma_2 N})}{\det^{\alpha/\gamma_1}(\imath H + \mathbf{1}_{\gamma_2 N})} = & \mathrm{Vol}(\mathrm{G}_N) \, \gamma_2^{-N-\beta N(N-1)/2+\alpha\beta/2} \\ & \times \int_{\mathbb{R}^N} \mathrm{d}[Y] \, \frac{|\Delta_N(Y)|^\beta \exp(\mathrm{tr}(\imath Y + \gamma_2 \mathbf{1}_N))}{\det^{\alpha\beta/2}(\imath Y + \gamma_2 \mathbf{1}_N)} \, , \end{aligned} \tag{D.5}$$

where $\mathrm{Vol}\,(\mathrm{G}_N)$ is the volume of the diagonalizing group G_N and depends on the particular choice of normalization of the Haar measure. Because it does not depend on the function we are integrating over, it can be determined in a simple way using a different weight function. The remaining eigenvalue integral reduces to a Selberg integral. To show this we use [40]

$$\begin{aligned}\int_{\mathbb{R}^N} \mathrm{d}[Y] \; \frac{|\Delta_N(Y)|^{2\kappa}}{\det^{l_a}(\imath Y + a\mathbf{1}_N)\det^{l_b}(\imath Y - b\mathbf{1}_N)} &= \frac{(2\pi)^N}{(a+b)^{(l_a+l_b)N-\kappa N(N-1)-N}} \\ \times \prod_{j=0}^{N-1} \frac{\Gamma(1+\kappa+j\kappa)\,\Gamma(l_a+l_b-(N+j-1)\kappa-1)}{\Gamma(1+\kappa)\,\Gamma(l_a-j\kappa)\,\Gamma(l_b-j\kappa)} \; . \end{aligned} \tag{D.6}$$

We take l_b to be a function of $b' = b + \gamma_2$ such that in the limit $b' \to \infty$, $l_b(b)/b' \to 1$ and $a = \gamma_2$. If we rescale both sides of Eq. (D.6) by a factor b'^{l_b} and perform the limit $b' \to \infty$, we obtain

$$\int_{\mathbb{R}^N} \mathrm{d}[Y] \; \frac{|\Delta_N(Y)|^{2\kappa} \exp \mathrm{tr}\,(\imath Y + \gamma_2 \mathbf{1}_N)}{\det^{l_a}(\imath Y + \gamma_2\mathbf{1}_N)} = (2\pi)^N \prod_{j=0}^{N-1} \frac{\Gamma(1+\kappa+j\kappa)}{\Gamma(1+\kappa)\,\Gamma(l_a-j\kappa)} \; . \tag{D.7}$$

It remains to compute the volume of G_N. This is achieved by choosing a weight which is most convenient for the calculation. We take a Gaussian distributed matrix H in the same symmetry class as the H in Eq. (D.1) and the flat measure to be

$$\mathrm{d}[H] \; = \prod_{i=1}^{N} \mathrm{d}H_{ii} \prod_{l=1}^{\beta}\prod_{i<j} \mathrm{d}H_{ij}^{l} \tag{D.8}$$

such that the group volume derives to

$$\mathrm{Vol}\,(\mathrm{G}_N) = \frac{\int \mathrm{d}[H] \;\; \exp\left(-\mathrm{tr}H^2\right)}{\int\limits_{-\infty}^{\infty} \mathrm{d}[X] \;\; |\Delta_N(x)|^{\beta} \exp\left(-\mathrm{tr}X^2\right)} = \sqrt{\pi}^{\beta N(N-1)/2} \prod_{j=1}^{N} \frac{\Gamma(1+\beta/2)}{\Gamma(1+j\beta/2)} \; . \tag{D.9}$$

If we set $A = \mathbf{1}_N$ on the right hand side of Eq. (D.4), it reduces to

$$I_{\alpha,N}(A) = C \; , \tag{D.10}$$

such that a comparison with Eq. (D.5) leads to

$$C = \sqrt{\frac{\pi}{\gamma_2}}^{\beta N(N-1)/2} \gamma_2^{\alpha\beta/2} \prod_{j=1}^{N} \frac{2\pi/\gamma_2}{\Gamma(\alpha+\beta/2-j\beta/2)} \; , \tag{D.11}$$

and therefore completes this section.

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